Appendix A

Useful properties of Legendre functions

The following expressions are mainly taken from Abramowitz & Stegun (1964) and Whittaker & Watson (1927).

Differential equation:

$$(1-x^2)\frac{\mathrm{d}^2 P_l^m}{\mathrm{d}x^2} - 2x\frac{\mathrm{d}P_l^m}{\mathrm{d}x} + \left[l(l+1) - \frac{m^2}{1-x^2}\right] P_l^m = 0 \tag{A.1}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[(1-x^2)\frac{\mathrm{d}P_l^m}{\mathrm{d}x}\right] + \left[l(l+1) - \frac{m^2}{1-x^2}\right] P_l^m = 0 \tag{A.2}$$

Legendre Polynomials: $P_l(x) = P_l^0(x)$. Explicit expressions for the first few cases:

$$\begin{aligned} P_0(x) &= 1 \\ P_1(x) &= x \\ P_1^1(x) &= -(1-x^2)^{1/2} \\ P_2(x) &= 1/2(3x^2-1) \end{aligned} \tag{A.3}$$

General expressions, for $m > 0$:

$$P_l(x) = \frac{1}{2^l l!}\frac{\mathrm{d}^l (x^2-1)^l}{\mathrm{d}x^l} \tag{A.4}$$

$$P_l^m(x) = (-1)^m (1-x^2)^{m/2}\frac{\mathrm{d}^m P_l(x)}{\mathrm{d}x^m} \tag{A.5}$$

$$P_m^m(x) = (-1)^m \frac{(2m)! 2^{-m}}{m!}(1-x^2)^{m/2} \tag{A.6}$$

Note that equation (A.4) shows that $P_l(x)$ is a polynomial of degree l. Also, equation (A.6) shows that

$$P_m^m(\cos\theta) = (-1)^m \frac{(2m)! 2^{-m}}{m!} \sin^m\theta \tag{A.7}$$

Expression for negative azimuthal order:

$$P_l^{-m}(\cos\theta) = \frac{(l-m)!}{(l+m)!} P_l^m(\cos\theta) \; . \tag{A.8}$$

Recursion relations:

$$(l-m+1)P_{l+1}^m(x) \;=\; (2l+1)xP_l^m(x) - (l+m)P_{l-1}^m(x) \tag{A.9}$$

$$(1-x^2)\frac{\mathrm{d}P_l^m}{\mathrm{d}x} \;=\; lxP_l^m(x) - (l+m)P_{l-1}^m(x) \tag{A.10}$$

$$x\frac{\mathrm{d}P_l}{\mathrm{d}x} - \frac{\mathrm{d}P_{l-1}}{\mathrm{d}x} \;=\; lP_l(x) \tag{A.11}$$

Integrals:

$$\int_{-1}^{1} P_l^m(x)P_{l'}^m(x)\mathrm{d}x = \delta_{ll'}\frac{(n+m)!}{(l+1/2)(l-m)!} \tag{A.12}$$

Asymptotic expansion, for $m \geq 0$, large l:

$$P_l^m(\cos\theta) = \frac{\Gamma(l+m+1)}{\Gamma(l+3/2)}(\frac{\pi}{2}\sin\theta)^{-1/2}\cos\left[(l+\frac{1}{2})\theta - \frac{\pi}{4} + \frac{m\pi}{2}\right] + \mathcal{O}(l^{-1}) \tag{A.13}$$