Appendix B

Effects of a perturbation on acoustic-mode frequencies

In several of the Problems we consider the effect on the frequencies of acoustic modes, as described by the asymptotics leading to the Duvall law, of various modifications (such as the perturbation in the gravitational potential, changes in the sound speed and rotation). These results all reflect a more general expression, which is derived here.

I start from the dispersion relation for a plane sound wave, with the addition of some perturbation $\delta_r a(r)$ which, as indicated, is assumed to be a function of r alone:

$$\omega^2 = c^2 |\mathbf{k}|^2 + \delta_r a(r) \tag{B.1}$$

(*cf.* eq. 3.55), where for simplicity I dropped "0" on equilibrium quantities. I now use that $|\mathbf{k}|^2 = k_r^2 + k_{\rm h}^2$, where $k_{\rm h}^2$ is given by equation (4.51). Therefore, k_r is given by

$$\begin{aligned} k_r &= \left(\frac{\omega^2}{c^2} - \frac{L^2}{r^2} - \frac{1}{c^2}\delta_r a \right)^{1/2} \\ &\simeq \frac{\omega}{c} \left[\left(1 - \frac{L^2 c^2}{\omega^2 r^2} \right)^{1/2} - \frac{1}{2\omega^2} \left(1 - \frac{L^2 c^2}{\omega^2 r^2} \right)^{-1/2} \delta_r a \right] , \end{aligned} \tag{B.2}$$

where in the last equality I assumed that $\delta_r a$ was small. The condition that we have a standing wave can be expressed as

$$\int_{r_{\rm t}}^{R} k_r {\rm d}r = (n + \alpha)\pi , \tag{B.3}$$

where, as usual, α takes into account the phase change at the surface. By substituting equation (B.2) we obtain

$$\frac{(n+\alpha)\pi}{\omega} \simeq \int_{r_{\rm t}}^{R} \left(1 - \frac{L^2 c^2}{\omega^2 r^2} \right)^{1/2} \frac{{\rm d}r}{c} - \frac{1}{2\omega^2} \int_{r_{\rm t}}^{R} \left(1 - \frac{L^2 c^2}{\omega^2 r^2} \right)^{-1/2} \delta_r a \frac{{\rm d}r}{c} . \tag{B.4}$$

If we neglect the term in $\delta_r a$ we obviously obtain the usual Duvall law, equation (7.1). Hence equation (B.4) shows the effect of the perturbation on the Duvall law.

We can now find the effect on the oscillation frequencies of the perturbation. I assume that the result is to change the frequency from ω to $\omega + \delta\omega$. Also it should be recalled that $\alpha = \alpha(\omega)$ in general depends on ω. By multiplying equation (B.4) by ω and perturbing it we obtain

$$\begin{aligned} \pi \frac{\mathrm{d}\alpha}{\mathrm{d}\omega} \delta\omega &= \delta\omega \int_{r_\mathrm{t}}^{R} \left(1 - \frac{L^2 c^2}{\omega^2 r^2}\right)^{1/2} \frac{\mathrm{d}r}{c} + \omega \int_{r_\mathrm{t}}^{R} \left(1 - \frac{L^2 c^2}{\omega^2 r^2}\right)^{-1/2} \frac{L^2 c^2}{\omega^2 r^2} \frac{\delta\omega}{\omega} \frac{\mathrm{d}r}{c} \\ &- \frac{1}{2\omega} \int_{r_\mathrm{t}}^{R} \left(1 - \frac{L^2 c^2}{\omega^2 r^2}\right)^{-1/2} \delta_r a \frac{\mathrm{d}r}{c} \ . \end{aligned} \tag{B.5}$$

From this we finally obtain

$$S \frac{\delta\omega}{\omega} \simeq \frac{1}{2\omega^2} \int_{r_\mathrm{t}}^{R} \left(1 - \frac{L^2 c^2}{\omega^2 r^2}\right)^{-1/2} \delta_r a \frac{\mathrm{d}r}{c} \ , \tag{B.6}$$

where

$$S = \int_{r_\mathrm{t}}^{R} \left(1 - \frac{L^2 c^2}{\omega^2 r^2}\right)^{-1/2} \frac{\mathrm{d}r}{c} - \pi \frac{\mathrm{d}\alpha}{\mathrm{d}\omega} \ . \tag{B.7}$$

This is the desired general expression.

It should be noticed that equation (B.6) has a very simple physical interpretation: Apart from the (generally small) term in $\mathrm{d}\alpha/\mathrm{d}\omega$ in S, the equation shows that the change in ω^2 is just a weighted average of $\delta_r a$, with the weight function

$$\mathcal{W}(r) = \frac{1}{c} \left(1 - \frac{L^2 c^2}{\omega^2 r^2}\right)^{-1/2} \ . \tag{B.8}$$

However, it is easily seen that $\mathcal{W}(r)\mathrm{d}r$ is just the sound travel time, corresponding to the radial distance $\mathrm{d}r$, along the ray describing the mode. Hence the weight in the average simply gives the time that the mode, regarded as a superposition of plane waves, spends in a given region of the star.

Example I. Effect of a change in sound speed: If the sound speed is changed from c to $c + \delta_r c$, the dispersion relation for sound waves can be written

$$\omega^2 = c^2 |\mathbf{k}|^2 + 2c\delta_r c |\mathbf{k}|^2 = c^2 |\mathbf{k}|^2 + 2\omega^2 \frac{\delta_r c}{c} \ . \tag{B.9}$$

Hence here $\delta_r a = 2\omega^2 \delta_r c / c$. Consequently, the frequency change is given by

$$S \frac{\delta\omega}{\omega} \simeq \int_{r_\mathrm{t}}^{R} \left(1 - \frac{L^2 c^2}{\omega^2 r^2}\right)^{-1/2} \frac{\delta_r c}{c} \frac{\mathrm{d}r}{c} \ . \tag{B.10}$$

It should be noted that in general a change to the model would result also in an intrinsic change $\delta\alpha$ in α. As a result, the general asymptotic expression for the frequency change [which can be obtained by including explicitly a term in the change in α when deriving equation (B.5) from equation (B.4)] is

$$S \frac{\delta\omega}{\omega} \simeq \int_{r_\mathrm{t}}^{R} \left(1 - \frac{L^2 c^2}{\omega^2 r^2}\right)^{-1/2} \frac{\delta_r c}{c} \frac{\mathrm{d}r}{c} + \pi \frac{\delta\alpha}{\omega} \ . \tag{B.11}$$

Thus we recover equation (7.145).

Example II. Effect of rotation: As discussed in Chapter 8 the dominant effect of rotation is purely geometrical. In a system rotating with the star, the dispersion relation is as usual $\omega_0^2 = c^2|\mathbf{k}|^2$; from equation (8.2) the dispersion relation in the inertial system is therefore, to lowest order in Ω,

$$\omega^2 = c^2|\mathbf{k}|^2 + 2m\omega\Omega \; , \tag{B.12}$$

where I assume that the rotation rate Ω depends on r alone. Hence here $\delta_r a(r) = 2m\omega\Omega(r)$, and we obtain the perturbation in the frequency caused by rotation as

$$S\delta\omega \simeq m \int_{r_\mathrm{t}}^{R} \left(1 - \frac{L^2 c^2}{\omega^2 r^2}\right)^{-1/2} \Omega(r) \frac{\mathrm{d}r}{c} \; . \tag{B.13}$$