Appendix C

Problems

The following problems have been used in courses on stellar pulsations in Aarhus. They originally appeared on the weekly problem sheets (Ugesedler), but are collected here for convenience and more general use. In a few cases reference is made to programming and plotting with IDL. Obviously any other convenient graphics package may be used instead.

The problems are collected in sections corresponding approximately to the sections of the main text, although the numbering has not been maintained.

C.1 Analysis of oscillation data

Problem 1.1:

Discrete Fourier transform. Observationally, the data are typically given at discrete times t_n. Hence the continuous Fourier transform considered in Section 2.2 is not immediately relevant. Here we consider the simple case where the data are uniformly spaced in time,

$$t_n = n\Delta t \; , \qquad n = 0, \ldots, N-1 \tag{C.1}$$

(this can in fact often be arranged). Then we define the discrete Fourier transform as

$$\hat{v}(\omega_j) = \frac{1}{N} \sum_{n=0}^{N-1} v(t_n) \exp(i\omega_j t_n) \; , \tag{C.2}$$

given at the discrete frequencies ω_j. For the moment we do not specify ω_j.

i) Find the discrete transform of the simple harmonic oscillator given in equation (2.19). Sketch the power $|\hat{v}(\omega_j)|^2$. Compare with the continuous transform.

A very efficient procedure for computing the discrete transform is the *Fast Fourier Transform* (FFT). This requires that the number of data points is a power 2^μ of 2, and provides the transform only at the frequencies

$$\omega_j = \frac{2\pi}{N\Delta t} j = \frac{2\pi}{T} j \; , \qquad j = 0, \ldots, N \; , \tag{C.3}$$

where $T = N\Delta t$ is the total duration of the observing run. For more details about the FFT, see also *Numerical Recipes* (Press *et al.* 1986).

ii) Sketch the power resulting from a FFT of the time string given by the harmonic oscillator in equation (2.19); for simplicity take the frequency ω_0 of the oscillator to be $2\pi j_0/(N\Delta t)$ for some integer j_0. Note that very little of the sinc function structure in the continuous transform is preserved.

Normally, the number of data points will evidently not be an integral power of 2. In that case, the data are extended by zeros up to a total number $N_1 = 2^\mu > N$.

iii) Consider the effect on the FFT of the simple harmonic oscillator of extending the data by zeros in this way. Show how this can be used to resolve the structure of the peak of the power spectrum (note that we do not need to take the smallest possible power of 2).

Note: In manipulating the Fourier transforms, it is useful to recall that

$$\cos x = \frac{1}{2}(e^{ix} + e^{-ix}) ;$$
$$\sum_{n=0}^{N} x^n = \frac{1 - x^{N+1}}{1 - x} . \tag{C.4}$$

Problem 1.2:

Multiperiodic harmonic oscillations. (This requires access to a computer with graphics facilities. Here reference is made to the graphics and data analysis package IDL, which is a very convenient tool for this type of exercise).

Consider a time string consisting of a sum of harmonic oscillators,

$$v(t) = \sum_{k=0}^{K} a_k \cos(\omega_k t - \delta_k) . \tag{C.5}$$

Take the amplitudes a_k and the phases δ_k to be uniformly distributed random numbers, between 0 and 1, and 0 and 2π, respectively. Also choose suitably "random" frequencies ω_k.

i) Plot $v(t)$ for a total time T that is substantially larger than the longest period $2\pi/\omega_k$, for several different numbers K of individual oscillations. Note how the appearance of the time string becomes increasingly "chaotic" as K is increased.

ii) Plot the power spectra of the time strings found under i), using IDL's FFT routine. Consider the extent to which the individual frequencies can be separated. How does this depend on the number of modes or the total duration of the time string?

A typical observed time string of solar oscillations (corresponding, for example, to observations made in light integrated over the solar surface) contains of order 50 modes. Clearly, separating them is no trivial task.

Problem 1.3:

A non-harmonic oscillator. A simple model of a large-amplitude pulsating star (*e.g.* a Cepheid) is provided by the following function

$$v(t) = a_0 \sin[\omega t + \alpha \sin(\omega t)] \; . \tag{C.6}$$

i) Show that $v(t)$ is periodic, with the period $\Pi = 2\pi/\omega$.

ii) Sketch (or get IDL to plot) $v(t)$ for $\omega = 1$, $a_0 = 1$, $\alpha = 0.5$.

iii) For small α, find a simple expression for $v(t)$ by expanding it in powers of α, including the term of order α^2. (If this is too hard, consider just the term of order α). Try to guess what the expansion to higher orders in α will look like, qualitatively.

iv) Use IDL to find the Fourier transform and power spectrum of $v(t)$ for some typical cases. Compare with the expansion obtained in iii).

Problem 1.4:

A double-mode non-harmonic oscillator. In Problem 1.3 we considered a simple model of a large-amplitude pulsating star, with the signal $v(t) = a_0 \sin[\omega t + \alpha \sin(\omega t)]$; the effect of the term in α is roughly to give a distortion to the phase which depends on the instantaneous amplitude of the oscillation. This distortion may result from non-linear effects near the surface of the star affecting what is essentially a harmonic oscillation in the stellar interior.

We can generalize this concept to multiperiodic oscillators. For simplicity, we consider a double-mode star, and assume that the signal can be written as

$$\begin{aligned} v(t) &= a_1 \sin[\omega_1 t + \alpha a_1 \sin(\omega_1 t) + \alpha a_2 \sin(\omega_2 t)] \\ &+ a_2 \sin[\omega_2 t + \beta a_1 \sin(\omega_1 t) + \beta a_2 \sin(\omega_2 t)] \; . \end{aligned} \tag{C.7}$$

i) For small α and β, expand $v(t)$ in terms of harmonic functions. Include enough terms to get a feel for what the form of a general term might be.

ii) Use IDL to plot the time series and the power spectrum for typical cases. (It may be a good idea to use a logarithmic scale for the power, suitably truncated, to show the presence of weak peaks.) Compare with the expansion in i).

iii) What would happen if the star were pulsating in several modes?

Problem 1.5:

Spatial analysis of oscillation observations. As mentioned briefly in Section 2.1, the analysis of observations of solar oscillations essentially proceeds by multiplying the observed Doppler velocity field $v(\theta, \phi, t)$ by different spherical harmonics $Y_{l_0}^{m_0}(\theta, \phi)$ and integrating over the area A of the solar disk that is observed. The result is a function $v_{l_0 m_0}(t)$ of time that predominantly contains contributions from modes with $l = l_0$, $m = m_0$. Had the integration been over the entire Sun, the orthogonality of the spherical harmonics would have given complete isolation of these modes. However, since we can only see part of the Sun other modes leak in and complicate the analysis.

To illustrate these effects, we take out the ϕ-dependent part of the analysis, by considering an oscillation of the form

$$v(t) = V_0 \cos(m\phi - \omega t) \; . \tag{C.8}$$

This signal is observed over the interval in longitude from $\phi = -\phi_{\rm c}$ to $\phi = \phi_{\rm c}$ (we take $\phi = 0$ to correspond to the centre of the disk). The analysis results in an average over the interval:

$$v_{m_0}(t) = \frac{1}{2\phi_{\rm c}} \int_{-\phi_{\rm c}}^{\phi_{\rm c}} v(t) \cos(m_0 \phi) {\rm d}\phi \; . \tag{C.9}$$

i) Write $v_{m_0}(t)$ as

$$v_{m_0}(t) = S_{m_0 m} V_0 \cos(\omega t) \tag{C.10}$$

and find an expression for the *spatial response function* $S_{m_0 m}$. Have we seen something similar before?

ii) What happens in the limits of very small $\phi_{\rm c}$; or $\phi_{\rm c} = \pi$? What does the latter case correspond to?

iii) Plot $S_{m_0 m}$ for some typical cases.

Note that this behaviour is in fact very similar to the behaviour of the corresponding response functions $S_{l_0 m_0 l m}$ for the real observations. In fact, the calculation carried out here is essentially the ϕ-part of the full integral over the observed area on the solar disk.

Problem 1.6:

Spatial response functions. We consider observations of solar oscillations through an aperture of radius d, in units of the radius of the solar disk, and centred on the disk.

i) Show that the velocity response function, defined in analogy with equation (2.7), is given by

$$S_l^{(v)}(d) = \frac{2\sqrt{2l+1}}{d^2} \int_{x_1}^{1} P_l(x) x^2 \mathrm{d}x \; , \tag{C.11}$$

where $x_1 = \sqrt{1-d^2}$.

ii) Using the explicit expressions and recursion relations for $P_l(x)$ in Appendix A, calculate $S_l^{(v)}(d)$ for $d = 1$ and $d = 0.5$, and for as many l-values as you can be bothered to consider. What is the effect of restricting the aperture?

iii) A more intelligent way to compute the responses can be obtained by deriving recursion relations for the functions

$$\mathcal{P}_l^k(x) = \int x^k \frac{\mathrm{d}P_l}{\mathrm{d}x} \mathrm{d}x \; , \qquad \mathcal{Q}_l^k(x) = \int x^k P_l(x) \mathrm{d}x \; , \tag{C.12}$$

based on the recursion relations for P_l and its derivative, as well as a little integration by parts. Try to see whether you can find a way of doing that, and make a more extensive computation for the cases considered in ii). It may help to look in Christensen-Dalsgaard & Gough (1982).

C.2 A little hydrodynamics

Problem 2.1:

Waves at a density discontinuity. In Section 3.3.3 a relation is derived for gravity waves on a free surface. It is interesting also to consider gravity waves at an interface where the density jumps discontinuously. Examples are waves in a glass with oil on top of water, or waves at the interface between a helium-rich core and a hydrogen-rich envelope.

- Consider a system consisting of an infinite layer of density ρ_1 on top of an infinite layer of density ρ_2, with $\rho_1 < \rho_2$. The assumptions are otherwise as in Section 3.3.3. By requiring that the vertical displacement and the pressure are continuous at the perturbed interface between the layers, show that the oscillation frequencies are given by

$$\omega^2 = \frac{\rho_2 - \rho_1}{\rho_2 + \rho_1} g_0 k \; . \tag{C.13}$$

Does this make sense in the limits $\rho_1 \to 0$ and $\rho_1 \to \rho_2$?

Problem 2.2:

Damping of simple sound waves. To illustrate the effects of non-adiabaticity, we consider the radiative damping of the simple sound waves discussed in Section 3.3.1.

Except where otherwise noted, the assumptions are the same as in that section.

At low density, we use Newton's law of cooling, equation (3.23), for the radiative cooling. Also, we assume the ideal gas law, so that in particular $\Gamma_1 = \Gamma_2 = \Gamma_3 = \gamma = 5/3$, and that there is no nuclear energy generation. Finally, we consider a perturbation in the form of a plane wave, as in equation (3.53).

i) Assume that the opacity κ is constant (*i.e.*, independent of ρ and T). Show that the relation between p' and ρ', on complex form, can be written

$$\frac{p'}{p_0} = \gamma_{\rm N} \frac{\rho'}{\rho_0} \; , \tag{C.14}$$

where

$$\gamma_{\rm N} = \gamma \phi_{\rm N} \; , \qquad \phi_{\rm N} = \frac{1 + \dfrac{3i}{4\gamma\omega\tau_{\rm N}}}{1 + \dfrac{i}{\omega\tau_{\rm N}}} \; , \tag{C.15}$$

and

$$\tau_{\rm N} = \frac{p_0}{4a\tilde{c}\kappa_0\rho_0(\gamma - 1)T_0^4} \tag{C.16}$$

is a characteristic time scale for cooling by Newton's law.

ii) Show that the dispersion relation for a plane sound wave is

$$\omega^2 = c_0^2 |\mathbf{k}|^2 \phi_{\rm N} \; , \tag{C.17}$$

where $c_0 = (\gamma p_0/\rho_0)^{1/2}$ is the adiabatic sound speed.

iii) Consider a wave where the wave number $\mathbf{k}$ is real. Show that the imaginary part of ω is negative and that therefore the wave is damped, *i.e.*, that the amplitude decreases with time. What is the physical explanation for the damping? (You may assume that the damping is weak, so that $\mathrm{Re}(\omega) \gg \mathrm{Im}(\omega)$.)

iv) As a more realistic case, assume that the opacity depends on ρ and T, such that $\kappa \propto \rho^a T^b$ where a and b are positive (this corresponds to conditions in stellar atmospheres where the opacity is dominated by H^- absorption). Find the expression for $\gamma_{\rm N}$ in this case, and verify that the wave is damped in this case also. What is the effect of the opacity variation on the magnitude of the damping rate?

v) Discuss the behaviour of the dispersion relation in the limits $\tau_{\rm N} \to \infty$ and $\tau_{\rm N} \to 0$. Consider also the behaviour corresponding to $\rho \to 0$ at fixed T (corresponding to conditions high in a stellar atmosphere).

We now consider the limit of high density, where the diffusion approximation, equation (3.22), can be used for the radiative flux. We still assume that equilibrium quantities are constant and consider a plane wave.

vi) Show that the perturbation in the radiative cooling rate is given by

$$(\mathrm{div}\,\mathbf{F})' = \frac{4a\tilde{c}T_0^4}{3\kappa_0\rho_0}|\mathbf{k}|^2\frac{T'}{T_0} \tag{C.18}$$

(remember that we assume that $\nabla T_0 = 0$).

vii) Hence show that equations (C.14) and (C.17) are still valid, if $\gamma_{\rm N}$ and $\phi_{\rm N}$ are replaced by $\gamma_{\rm F}$ and $\phi_{\rm F}$, where

$$\gamma_{\rm F} = \gamma\phi_{\rm F} \; , \qquad \phi_{\rm F} = \frac{1 + \dfrac{i}{\gamma\omega\tau_{\rm F}}}{1 + \dfrac{i}{\omega\tau_{\rm F}}} \; , \qquad \tau_{\rm F} = \frac{3\kappa_0\rho_0 p_0}{4a\tilde{c}(\gamma-1)T_0^4|\mathbf{k}|^2} \; . \tag{C.19}$$

Compare this expression for the time scale $\tau_{\rm F}$ for radiative diffusion with the estimate given in Section 3.1.4.

viii) Show that this also results in damping of a sound wave. Discuss the physics of the damping.

Finally, we consider the effects of nuclear energy generation on the waves. We assume that the energy generation rate $\epsilon \propto \rho T^n$, and that radiative effects can be neglected.

ix) Show that in this case the relation between p' and ρ' can be written

$$\frac{p'}{p_0} = \gamma_\epsilon\frac{\rho'}{\rho_0} \; , \tag{C.20}$$

where

$$\gamma_\epsilon = \gamma\phi_\epsilon \; , \qquad \phi_\epsilon = \frac{1 - \dfrac{i(n-2)}{n\gamma\omega\tau_\epsilon}}{1 - \dfrac{i}{\omega\tau_\epsilon}} \; , \tag{C.21}$$

and

$$\tau_\epsilon = \frac{p_0}{n\rho_0\epsilon_0(\gamma-1)} \tag{C.22}$$

is a characteristic time scale for nuclear heating.

x) Show that this leads to excitation of the sound wave, such that its amplitude grows with time. Discuss the physics of the excitation.

xi) Discuss qualitatively the case where both nuclear energy generation and radiative damping, in the diffusion approximation, are taken into account. For which waves might the overall effect be an excitation?

Problem 2.3:

Dispersion relation for sound waves in a gravitating fluid. We consider acoustic waves in a homogeneous system, as in Section 3.3.1, but include the effect of self-gravity on the wave. Hence in the equation of motion, the term $\rho_0 \nabla \Phi'$ should be included on the right-hand side, but we continue to neglect the equilibrium gravitational acceleration.

i) Show that equation (3.51) should be replaced by

$$\frac{\partial^2 \rho'}{\partial t^2} = c_0^2 \nabla^2 \rho' + 4\pi G \rho_0 \rho' \ . \tag{C.23}$$

ii) We assume a plane-wave solution, on the form given in equation (3.53). Show that the dispersion relation is

$$\omega^2 = c_0^2 \, |\mathbf{k}|^2 - 4\pi G \rho_0 \ . \tag{C.24}$$

iii) When the frequency ω obtained from equation (C.24) is imaginary, the perturbation either grows or decays exponentially with time. Show that this occurs when the wavelength λ of the wave satisfies $\lambda > \lambda_{\rm crit}$ and find $\lambda_{\rm crit}$. Compare with equation (10.4) of *Lecture Notes on Stellar Structure and Evolution.*

C.3 Properties of solar and stellar oscillations

Problem 3.1:

Rays of sound waves. Ray theory is a very powerful tool for understanding the propagation of waves in a medium where conditions vary slowly. It is entirely analogous to the study of the propagation of light rays. An important example is its use to describe the properties of acoustic oscillations in the Sun. We consider an acoustic wave that propagates in the $(x - z)$ plane where the sound speed c depends on z but not x. The wave is described by a wave vector $\mathbf{k} = (k_x, k_z)$ and has a given frequency ω; one may think of it as being excited with a fixed frequency at some point in the region. Clearly, ω and $\mathbf{k}$ satisfy the dispersion relation (3.55). The $x-$component k_x of $\mathbf{k}$ is fixed, whereas the $z-$component k_z depends on z.

i) Show from equation (3.55) that

$$k_z = \left(\frac{\omega^2}{k_x^2 c^2} - 1 \right)^{1/2} k_x \; . \tag{C.25}$$

The propagation of the ray is parallel to the wave vector. Hence we can describe the position (x, z) of a point on the ray by the equations

$$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}s} &= k_x \\ \frac{\mathrm{d}z}{\mathrm{d}s} &= k_z \; , \end{aligned} \tag{C.26}$$

where s is a suitably chosen measure of position along the ray. From these equations we obtain

$$\frac{\mathrm{d}x}{\mathrm{d}z} = \frac{k_x}{k_z} \; , \tag{C.27}$$

which determines the shape of the ray.

ii) Let z measure depth beneath some surface, and assume that $c(z)^2$ increases linearly with z,

$$c(z)^2 = c_0^2 + Az \; , \tag{C.28}$$

where c_0 and A are constants (this corresponds approximately to conditions near a stellar surface). Try to sketch, qualitatively, the behaviour of the ray.

iii) Find the solution to the equation for the ray.

iv) What happens at a horizontal interface where c jumps from a value c_1 to a value c_2? Have you seen anything like that before?

Problem 3.2:

Trapping of g modes. We consider g modes trapped near a maximum in the buoyancy frequency, corresponding to the steep gradient in composition outside a convective core. Specifically, we make the following assumptions:

- $S_l/\omega \gg 1$.
- N^2 is only non-zero in a narrow interval $[r_1, r_2]$:

$$N^2 = \begin{cases} 0 & \text{for } r < r_1 \\ N_\mathrm{m}^2 & \text{for } r_1 \le r \le r_2 \\ 0 & \text{for } r_2 < r \; . \end{cases} \tag{C.29}$$

- The region considered is so thin that we can make the plane-parallel assumption, replacing $l(l+1)/r^2$ by $k_{\rm h}^2$ which we take to be constant.
- The oscillations are described by the simplified equations (5.20) and (5.21).

i) Show under these assumptions that ξ_r approximately satisfies

$$\frac{{\rm d}^2 \xi_r}{{\rm d}r^2} = -k_{\rm h}^2 \left(\frac{N^2}{\omega^2} - 1 \right) \xi_r \; . \tag{C.30}$$

ii) Show that the solution to equation (C.30), corresponding to modes trapped in the region considered, have the form

$$\xi_r(r) = \begin{cases} A \exp[k_{\rm h}(r - r_1)] & \text{for } r < r_1 \\ B_1 \cos[\beta k_{\rm h}(r - r_1)] + B_2 \sin[\beta k_{\rm h}(r - r_1)] & \text{for } r_1 \le r \le r_2 \\ C \exp[-k_{\rm h}(r - r_2)] & \text{for } r_2 < r \; , \end{cases} \tag{C.31}$$

where

$$\beta^2 = \left(\frac{N_{\rm m}^2}{\omega^2} - 1 \right) \; . \tag{C.32}$$

iii) The solution must satisfy continuity of ξ_r and ${\rm d}\xi_r/{\rm d}r$. Show that this leads to the following relation which implicitly determines ω:

$$\tan(\beta \Delta) = \frac{2\beta}{\beta^2 - 1} \; , \tag{C.33}$$

where $\Delta = k_{\rm h}(r_2 - r_1)$.

iv) Sketch the left-hand side and the right-hand side of equation (C.33) as a function of β and argue that the equation has an infinite number of solutions β_n, corresponding to the frequencies ω_n, $n = 1, 2, \ldots$.

v) Assume Δ to be very large. Show that the lowest-order modes satisfy

$$\beta \Delta \simeq n\pi \; , \tag{C.34}$$

and hence

$$\omega^2 \simeq \frac{N_{\rm m}^2}{1 + (n\pi/\Delta)^2} \; . \tag{C.35}$$

vi) Sketch (or plot) the solution $\xi_r(r)$ in this approximation for the first few values of n.

vii) Does equation (C.35) look familiar? (Hint: Relate $r_2 - r_1$ to the vertical wavelength, and hence the vertical wavenumber k_r, of the mode.)

viii) Consider the opposite extreme of very high-order modes. Show that $\omega_n \to 0$ for $n \to \infty$, and that consequently the frequencies are still given by equation (C.35). Show also that the periods Π_n of pulsation may be approximated by

$$\Pi_n \simeq \Pi_0 n \; , \qquad \Pi_0 = \frac{2\pi^2}{N_{\rm m}\Delta} \; . \tag{C.36}$$

Thus the periods increase linearly with n.

ix) Try to solve the dispersion relation (C.33) numerically, to obtain $\omega_n/N_{\rm m}$ as a function of Δ, and make plots of the corresponding eigenfunctions.

Problem 3.3:

A simple example of avoided crossings. Avoided crossings play a major role in understanding the properties of spectra of stellar oscillations. Here we analyze a very simple physical system that exhibits this behaviour.

Consider two coupled oscillators, with a time dependence described by $y_1(t)$ and $y_2(t)$, and satisfying the differential equations

$$\begin{aligned} \frac{{\rm d}^2 y_1}{{\rm d}t^2} &= -\omega_1(\lambda)^2 y_1 + \alpha y_2 \\ \frac{{\rm d}^2 y_2}{{\rm d}t^2} &= -\omega_2(\lambda)^2 y_2 + \alpha y_1 \; . \end{aligned} \qquad \text{(C.37)}$$

Here α is a coupling parameter, which we assume to be constant. In the absence of coupling (*i.e.*, for $\alpha = 0$) the oscillators have frequencies ω_1 and ω_2 which, as indicated, depend on a parameter λ. We assume that at $\lambda = \lambda_0$ the two uncoupled oscillators cross, *i.e.*, $\omega_1(\lambda_0) = \omega_2(\lambda_0)$

i) Show that the system in equation (C.37) has solutions of the form

$$\left\{ \begin{array}{c} y_1(t) \\ y_2(t) \end{array} \right\} = \left\{ \begin{array}{c} c_1 \\ c_2 \end{array} \right\} \exp(-i\omega t) \; , \qquad \text{(C.38)}$$

where the frequencies are given by

$$\omega_\pm^2 = \frac{1}{2}(\omega_1^2 + \omega_2^2) \pm \frac{1}{2}\left[(\omega_1^2 - \omega_2^2)^2 + 4\alpha^2\right]^{1/2} \; . \qquad \text{(C.39)}$$

ii) Discuss the behaviour of $\omega_\pm$, as functions of λ, far from λ_0 (in the sense that $|\omega_1^2(\lambda) - \omega_2^2(\lambda)| \gg \alpha$) and at $\lambda = \lambda_0$.

To analyze the behaviour of the system in more detail and find the coefficients, we simplify the expressions by assuming that $\omega_1^2 = 1$, $\omega_2^2 = \lambda$.

iii) Make a plot of $\omega_\pm^2(\lambda)$. Show that the two solutions are the two branches of a hyperbola.

iv) Find the coefficients $\{c_1^{(\pm)}(\lambda), c_2^{(\pm)}(\lambda)\}$, normalized such that $(c_1^{(\pm)})^2 + (c_2^{(\pm)})^2 = 1$. What is their behaviour far from the avoided crossing? And at the point of closest approach of the frequencies? Make a plot of the coefficients.

Problem 3.4:

A perturbation treatment of avoided crossings. As shown by von Neuman & Wigner (1929) it is possible to describe the behaviour of the eigenfrequencies near an avoided crossing by a simple extension of the usual first-order perturbation analysis. This Problem reproduces von Neuman & Wigner's analysis.

We consider the eigenvalue problem

$$\mathcal{F}\xi = \sigma\xi \; . \tag{C.40}$$

Here we express the operator $\mathcal{F}$ as

$$\mathcal{F} = \mathcal{F}_0 + \lambda\,\delta\mathcal{F} \; , \tag{C.41}$$

where $\lambda\,\delta\mathcal{F}$ is a small perturbation. (We use the real parameter λ to vary the perturbation; $\delta\mathcal{F}$ is a fixed operator.) We assume that both $\mathcal{F}_0$ and $\delta\mathcal{F}$ are symmetric operators. The eigenvalues and eigenvectors for the unperturbed operator $\mathcal{F}_0$ are given by

$$\mathcal{F}_0\xi_i^{(0)} = \sigma_i^{(0)}\xi_i^{(0)} \; , \qquad i = 1, 2, \dots \; . \tag{C.42}$$

Here the $\xi_i^{(i)}$ are taken to be normalized, $\|\xi_i^{(i)}\|^2 = \; < \xi_i^{(i)}, \xi_i^{(i)} > \; = 1$. We assume that $\sigma_1^{(0)}$ and $\sigma_2^{(0)}$ are very close together and neglect the remaining eigenvalues and eigenvectors in the following.

We let $\sigma(\lambda)$ be the eigenvalue of the perturbed operator, and assume that the corresponding eigenvector $\xi(\lambda)$ can be expressed as

$$\xi(\lambda) = c_1(\lambda)\xi_1^{(0)} + c_2(\lambda)\xi_2^{(0)} \; . \tag{C.43}$$

i) By substituting equation (C.43) into equation (C.40), and taking the scalar product of the resulting equation with $\xi_1^{(0)}$ and $\xi_2^{(0)}$, show that the eigenvalues are

$$\begin{aligned}\sigma_\pm(\lambda) \; = \; & \frac{1}{2}\left[\sigma_1^{(0)} + \sigma_2^{(0)} + \lambda(< \delta\mathcal{F} >_{11} + < \delta\mathcal{F} >_{22})\right] \\ & \pm\frac{1}{2}\left\{\left[\sigma_1^{(0)} - \sigma_2^{(0)} + \lambda(< \delta\mathcal{F} >_{11} - < \delta\mathcal{F} >_{22})\right]^2 + 4\lambda^2| < \delta\mathcal{F} >_{12} |^2\right\}^{1/2} \; ,\end{aligned} \tag{C.44}$$

where

$$< \delta\mathcal{F} >_{ij} = < \xi_i^{(0)}, \delta\mathcal{F}\xi_j^{(0)} > \; , \qquad i, j = 1, 2 \; . \tag{C.45}$$

ii) Sketch the behaviour of the eigenvalues. Show that the minimum separation between $\sigma_+(\lambda)$ and $\sigma_-(\lambda)$ is

$$\Delta\sigma_{\min} = \Delta\sigma_0 \frac{2| < \delta\mathcal{F} >_{12} |}{[(< \delta\mathcal{F} >_{11} - < \delta\mathcal{F} >_{22})^2 + 4| < \delta\mathcal{F} >_{12} |^2]^{1/2}} \; , \tag{C.46}$$

where $\Delta\sigma_0 = \sigma_2^{(0)} - \sigma_1^{(0)}$, and occurs at

$$\lambda = \lambda_{\rm min} = \frac{\Delta\sigma_0(< \delta\mathcal{F} >_{11} - < \delta\mathcal{F} >_{22})}{(< \delta\mathcal{F} >_{11} - < \delta\mathcal{F} >_{22})^2 + 4| < \delta\mathcal{F} >_{12} |^2} \; . \qquad \text{(C.47)}$$

iii) Compare this solution with the eigenfrequencies obtained in Problem 3.3.

iv) Show that if the coupling term $< \delta\mathcal{F} >_{12}$ is zero, equation (C.44) reduces to the usual expression for the change in the eigenvalue induced by a perturbation, equation (5.73).

v) Discuss the behaviour of the coefficients $c_{1,2}^{(\pm)}(\lambda)$ associated with the eigenvalues $\sigma_\pm(\lambda)$ in the limit where $|\Delta\sigma_{\rm min}/\Delta\sigma_0| \ll 1$.

Note that if the coupling term $< \delta\mathcal{F} >_{12}$ vanishes, the avoided crossing is replaced by a true crossing of the eigenvalues. It may be shown that this is the case for the eigenvalue problem in equation (5.56) if we consider two eigenfunctions $\boldsymbol{\xi}_1^{(0)}$ and $\boldsymbol{\xi}_2^{(0)}$ with different values of the degree l. This is the reason why the curves for $l = 0$ and $l = 1$, for instance, cross in Figure 5.14.

Problem 3.5:

The perturbation of the stellar surface. The displacement given in equation (4.40) shows how each part of the star is moved by the oscillation. We take $t = 0$.

i) Consider the behaviour in the equatorial plane. Sketch or plot the perturbed surface for $(l, m) = (0, 0), (1, 1), (2, 2)$ using the expressions in the notes on the Legendre functions. Consider different values of the ratio $\xi_{\rm h}/\xi_r$.

ii) Repeat i), but in the plane passing through the pole, at $\phi = 0$.

C.4 Asymptotic theory of stellar oscillations

Problem 4.1:

A simple derivation of the Duvall law. Equation (7.39) can be justified rather more simply than by going through the full JWKB analysis (even though the arguments are essentially equivalent). We start from the dispersion relation for a plane sound wave, neglecting self-gravity (equation 3.55):

$$\omega^2 = c_0^2 |\mathbf{k}|^2 \; , \qquad \text{(C.48)}$$

and write $|\mathbf{k}|^2 = k_r^2 + k_{\rm h}^2$. For a wave corresponding to a mode of oscillation, $k_{\rm h}$ is given by equation (4.51).

i) Dropping subscripts “0” on equilibrium quantities, show that

$$k_r^2 = \frac{\omega^2}{c^2} - \frac{L^2}{r^2} \; , \tag{C.49}$$

where $L^2 = l(l+1)$.

ii) Argue that the condition for a standing wave (*i.e.*, a mode of oscillation) is roughly that

$$\int_{r_{\rm t}}^{R} k_r {\rm d}r = n\pi \; . \tag{C.50}$$

A more careful analysis shows that n in equation (C.49) should be replaced by $n + \alpha$, where α takes care of the behaviour near the lower turning point $r_{\rm t}$ and the surface.

iii) Use equation (C.50), modified in this way, and equation (C.49) to derive equation (7.39).

Problem 4.2:

The effect of the gravitational potential perturbation on p-mode frequencies. By combining the results of Problems 2.3 and 4.1, we can estimate the error made in the Cowling approximation.

i) Repeat the analysis in Problem 4.1, but using the dispersion relation (C.24) (*cf.* Problem 2.3) for a gravitating fluid, to obtain the following modified form of equation (7.39):

$$\omega \int_{r_{\rm t}'}^{R} \left(1 - \frac{L^2 c^2}{\omega^2 r^2} + \frac{4\pi G \rho}{\omega^2} \right)^{1/2} \frac{{\rm d}r}{c} \simeq \pi(n+\alpha) \; . \tag{C.51}$$

ii) The last term in the bracket in equation (C.51) arises from the effect of the perturbation in the gravitational potential. Since this term is generally small, we can expand the bracket. Show that the result may be written as

$$\frac{\pi(n+\alpha)}{\omega} \simeq \int_{r_{\rm t}}^{R} \left(1 - \frac{L^2 c^2}{\omega^2 r^2} \right)^{1/2} \frac{{\rm d}r}{c} + \frac{2\pi G}{\omega^2} \int_{r_{\rm t}}^{R} \rho \left(1 - \frac{L^2 c^2}{\omega^2 r^2} \right)^{-1/2} \frac{{\rm d}r}{c} \; , \tag{C.52}$$

with the usual definition of $r_{\rm t}$.

We note that, as $r_{\rm t}$ is a function of ω/L, equation (C.52) may be written as

$$\frac{\pi(n+\alpha)}{\omega} = F\left(\frac{\omega}{L}\right) + \frac{1}{\omega^2} F_\Phi \left(\frac{\omega}{L}\right) \; , \tag{C.53}$$

where the functions $F(w)$ and $F_\Phi(w)$ are defined by equation (C.52). This is a generalization of equation (7.40).

From equation (C.52) we can derive an approximate expression for the difference $\delta\omega^{(\Phi)} = \omega^{(F)} - \omega^{(C)}$ between the frequency $\omega^{(F)}$ obtained taking the perturbation in the gravitational potential into account, and the frequency $\omega^{(C)}$ obtained in the Cowling approximation.

iii) Show by expanding equation (C.52) in $\delta\omega^{(\Phi)}$, including only the linear term, and neglecting a small term arising from the frequency-dependence of α, that

$$\delta\omega^{(\Phi)} \simeq -\frac{1}{\omega} \frac{2\pi G \int_{r_\mathrm{t}}^R \rho \left(1 - \frac{L^2 c^2}{\omega^2 r^2}\right)^{-1/2} \frac{\mathrm{d}r}{c}}{\int_{r_\mathrm{t}}^R \left(1 - \frac{L^2 c^2}{\omega^2 r^2}\right)^{-1/2} \frac{\mathrm{d}r}{c}} . \tag{C.54}$$

Thus the frequency change induced by the gravitational potential perturbation depends on an average of the density structure of the equilibrium model, over the region where the mode is trapped.

Problem 4.3:

Effects of a change to the model. We assume that the oscillation frequencies are given by the Duvall law, equation (7.104), where $\alpha = \alpha(\omega)$ is a function of frequency. Consider the case where the structure of the equilibrium model is changed (keeping the surface radius R fixed) such that $c(r)$ is replaced by $c(r) + \delta_r c(r)$, and $\alpha(\omega)$ is replaced by $\alpha(\omega) + \delta\alpha(\omega)$ (note that δ as used here should not be confused with the use of δ elsewhere to denote the Lagrangian perturbation). As a result of these changes, the eigenfrequency ω_{nl} is changed to $\omega_{nl} + \delta\omega_{nl}$.

i) Show that $\delta\omega_{nl}$ is given by

$$S_{nl} \frac{\delta\omega_{nl}}{\omega_{nl}} \simeq \int_{r_t}^R \left(1 - \frac{L^2 c^2}{r^2 \omega_{nl}^2}\right)^{-1/2} \frac{\delta_r c}{c} \frac{\mathrm{d}r}{c} + \pi \frac{\delta\alpha}{\omega_{nl}} , \tag{C.55}$$

where

$$S_{nl} = \int_{r_t}^R \left(1 - \frac{L^2 c^2}{r^2 \omega_{nl}^2}\right)^{-1/2} \frac{\mathrm{d}r}{c} - \pi \frac{\mathrm{d}\alpha}{\mathrm{d}\omega} . \tag{C.56}$$

ii) Sketch the behaviour of the weight function

$$\mathcal{W} = \frac{1}{c} \left(1 - \frac{L^2 c^2}{r^2 \omega_{nl}^2}\right)^{-1/2} \tag{C.57}$$

for a typical increase of the sound speed with depth (you may, for example, assume that c^2 increases roughly linearly with depth). What is the effect of the singularity at $r = r_t$?

iii) Discuss the physical interpretation of $\mathcal{W}$ (try to think in terms of travel time for sound waves).

iv) Having made it through this derivation, go back and reconsider Problem 4.2 on the effect of the perturbation in the gravitational potential on the oscillation frequencies.

Problem 4.4:

The effect of discontinuities on oscillation frequencies. Sharp features in the stellar model introduce characteristic oscillations in the frequencies as a function of mode order. An example was encountered in Section 7.7.3 where we discussed the effect on the phase function $\mathcal{H}_2(\omega)$ of the rapid variation in Γ_1 in the second helium ionization zone. Another important example is the effect of the boundaries of convective regions. Here we discuss the effects of such features in a very simple way. The analysis is based on Appendix B of Monteiro, Christensen-Dalsgaard & Thompson (1994; but see the end of this problem for a correction to that paper). We assume that the oscillations are described by equation (7.90).

i) At the base of a convective envelope the temperature gradient $\nabla \equiv \mathrm{d}\ln T/\mathrm{d}\ln p$ goes from being adiabatic in the convection zone to being radiative below it. The transition occurs very abruptly, in such a way that ∇ is continuous but its gradient $\mathrm{d}\nabla/\mathrm{d}r$ is essentially discontinuous (see also *Lecture Notes on Stellar Structure and Evolution*, Fig. 6.3a). Argue that as a result ω_c^2 in equation (7.90) is discontinuous, whereas the other terms are continuous.

ii) Certain simplified models of convective overshoot predict that ∇ is discontinuous at the edge of the convective region. Argue that in this case ω_c^2 has a δ-function singularity at the convection-zone boundary.

The results of i) and ii) suggests that we consider an equation of the form

$$\frac{\mathrm{d}^2 Y}{\mathrm{d}x^2} + [\omega^2 - V^2(x)]Y(x) = 0 \ , \qquad \text{(C.58)}$$

on the interval $[0, x_t]$, where the "potential" $V(x)$ has either a discontinuity or a δ-function behaviour at some location in $[0, x_t]$. The eigenfrequencies ω are determined by imposing the boundary conditions

$$Y(0) = Y(x_t) = 0 \ . \qquad \text{(C.59)}$$

iii) As a reference case we consider the potential being constant everywhere, $V = V_{\rm a}$, say. Show that the corresponding eigenfrequencies satisfy the following dispersion relation

$$\omega_0^2 - V_{\rm a}^2 = \left(\frac{n\pi}{x_{\rm t}}\right)^2 , \tag{C.60}$$

n being an integer corresponding to the (*number of zeros* + 1) of the eigenfunction.

iv) To illustrate the effects of a discontinuity we consider the following modified potential:

$$V_1(x) = \begin{cases} V_{\rm b} & \text{for } 0 \le x < \alpha_1 x_{\rm t} \\ V_{\rm a} & \text{for } \alpha_1 x_{\rm t} \le x < x_{\rm t} \ . \end{cases} \tag{C.61}$$

Show that Y and $\mathrm{d}Y/\mathrm{d}x$ are everywhere continuous. Hence, imposing the same boundary conditions as before, show that ω satisfies the dispersion relation

$$\tan(\Lambda_{\rm b}\alpha_1 x_{\rm t}) = -\frac{\Lambda_{\rm b}}{\Lambda_{\rm a}} \tan[\Lambda_{\rm a}(1-\alpha_1)x_{\rm t}] \ , \tag{C.62}$$

where

$$\Lambda_{\rm a} = \left(\omega^2 - V_{\rm a}^2\right)^{1/2} , \qquad \Lambda_{\rm b} = \left(\omega^2 - V_{\rm b}^2\right)^{1/2} . \tag{C.63}$$

v) To study the effect of a δ-function singularity, consider the following modified potential:

$$V_2^2(x) = V_{\rm a}^2 + A_\delta \delta(x - \alpha_2 x_{\rm t}) \ . \tag{C.64}$$

Show that $Y(x)$ is still everywhere continuous, whereas $\mathrm{d}Y/\mathrm{d}x$ satisfies the following jump condition at $x = \alpha_2 x_{\rm t}$:

$$\left.\frac{\mathrm{d}Y}{\mathrm{d}x}\right|_{\alpha_2 x_{\rm t}+} - \left.\frac{\mathrm{d}Y}{\mathrm{d}x}\right|_{\alpha_2 x_{\rm t}-} = A_\delta Y(\alpha_2 x_{\rm t}) \ . \tag{C.65}$$

(Hint: integrate the differential equation (C.58) across the $x = \alpha_2 x_{\rm t}$.) Hence derive the following dispersion relation for ω:

$$\tan(\Lambda_{\rm a}\alpha_2 x_{\rm t}) = -\frac{\tan[\Lambda_{\rm a}(1-\alpha_2)x_{\rm t}]}{1 + A_\delta \Lambda_{\rm a}^{-1} \tan[\Lambda_{\rm a}(1-\alpha_2)x_{\rm t}]} \ . \tag{C.66}$$

We now consider the discontinuity or the singularity as small perturbations on a high-order mode. Specifically, we assume that $\omega_0 \gg V_{\rm a}$; also, writing $\omega = \omega_0 + \delta\omega$, and $\delta V^2 = V_{\rm a}^2 - V_{\rm b}^2$, we assume that $|\delta\omega| \ll \omega_0$, $|\delta V^2| \ll V_{\rm a}^2$, and that $|A_\delta| \ll V_{\rm a}$.

vi) Show, by expanding equation (C.62) in terms of the small quantities, that in the case of a discontinuity the frequency change has a periodic component which is approximately given by

$$\delta\omega_{\rm p1} \sim \frac{\delta V^2}{4 x_{\rm t} \omega_0^2} \sin(2\Lambda_{\rm a}\alpha_1 x_{\rm t}) \ . \tag{C.67}$$

vii) Show, by expanding equation (C.66) in terms of the small quantities, that in the case of a δ-function singularity the frequency change has a periodic component which is approximately given by

$$\delta\omega_{\rm p2} \sim \frac{A_\delta}{2x_{\rm t}\omega_0} \cos(2\Lambda_{\rm a}\alpha_2 x_{\rm t}) \; . \tag{C.68}$$

Note that in both cases the analysis predicts a frequency perturbation which oscillates as a function of the unperturbed frequency ω_0. With the assumption that $\omega_0 \gg V_{\rm a}$ the "frequency" of this oscillation is approximately $2\omega_0\alpha_i x_{\rm t}$; hence it measures the location of the discontinuity or singularity. The two cases differ in the frequency dependence of the amplitude of the oscillation (ω_0^{-2} for a discontinuity, ω_0^{-1} for a singularity) and in the phase of the oscillation. This in principle allows a determination of the nature of the sharp feature. As discussed by Monteiro *et al.* an analysis of this nature allows testing for the presence of overshoot below the solar convection zone.

viii) (Optional) Some stars have growing convective cores during parts of the core hydrogen burning phase. Argue that this leads to a discontinuity in density. Try to carry out a similar analysis for this case.

Erratum to Monteiro et al. (1994): As found by a student in the course on Stellar Pulsations in 2012 there is a misprint in equation (B13). The correct equation is

$$\tan[\Lambda_a(\alpha_2\tau_t)] = -\frac{\tan[\Lambda_a(\tau_t - \alpha_2\tau_t)]}{1 + A_\delta(\omega^2 - V_a^2)^{-1/2}\tan[\Lambda_a(\tau_t - \alpha_2\tau_t)]}$$

C.5 Rotation and stellar oscillations

Problem 5.1:

Asymptotic description of rotational splitting. We can derive the asymptotic expression for the rotational splitting of p-mode frequencies very simply from the plane-wave treatment in Section 3.3.1. We neglect the Coriolis force, so that the only change in the equation of motion is the addition of the term $-2m\omega\Omega\rho_0\boldsymbol{\delta}\mathbf{r}$ on the right-hand side (*cf.* eq. 8.25). This is treated as a perturbation to the Duvall relation, equation (7.1), in much the same way as the analysis of the effect of the perturbation in the gravitational potential in Problem 4.2.

i) Show that the dispersion relation for plane sound waves is changed to

$$\omega^2 = c^2|\mathbf{k}|^2 + 2m\omega\Omega \; . \tag{C.69}$$

ii) We assume that the rotation rate $\Omega = \Omega(r)$ is a function of r alone, and that it is small. Show that the modified Duvall relation can be written

$$\pi\frac{n+\alpha}{\omega} = \int_{r_t}^{R}\left(1 - \frac{L^2c^2}{\omega^2 r^2}\right)^{1/2}\frac{\mathrm{d}r}{c} - \frac{m}{\omega}\int_{r_t}^{R}\left(1 - \frac{L^2c^2}{\omega^2 r^2}\right)^{-1/2}\Omega(r)\frac{\mathrm{d}r}{c}\ . \quad \text{(C.70)}$$

iii) Hence show that the effect of rotation is to produce a frequency shift $\delta\omega$ given by

$$S\,\delta\omega \simeq m\int_{r_t}^{R}\left(1 - \frac{L^2c^2}{r^2\omega^2}\right)^{-1/2}\Omega(r)\frac{\mathrm{d}r}{c}\ , \quad \text{(C.71)}$$

where

$$S = \int_{r_t}^{R}\left(1 - \frac{L^2c^2}{r^2\omega^2}\right)^{-1/2}\frac{\mathrm{d}r}{c} - \pi\frac{\mathrm{d}\alpha}{\mathrm{d}\omega}\ . \quad \text{(C.72)}$$

Does this look familiar?

Problem 5.2:

Asymptotic inversion of rotational splitting. Given the result of Problem 5.1, and the treatment of the asymptotic sound-speed inversion in Section 7.7.2, find a method for inverting observed rotational splittings to determine the rotation rate $\Omega(r)$, assuming it to be a function of r alone.

C.6 Excitation and damping of stellar oscillations

Problem 6.1:

Stochastic excitation of oscillations. It is generally believed that the observed modes of solar oscillation are damped. This has been indicated by a number of calculations, although there have been reports to the contrary also. If the modes are in fact stable, their most likely cause is the turbulent convection near the solar surface. In the uppermost part of the convection zone the velocity of the convective elements gets close to the sound speed. Such elements are quite efficient at emitting acoustic noise, and the noise excites the normal modes of the system. Essentially similar processes are responsible for the generation of the notes of wind instruments, including an organ.

In this Problem we do not go into the physical details of this excitation process, but instead concentrate on one aspect: its stochastic nature. The excitation of each mode is caused by the effect of a very large number of essentially uncorrelated convective elements. Consequently, the effective excitation force is a random function of time. Here we model the process through the behaviour of a damped harmonic oscillator that is excited by a random forcing. It may in fact be shown that this is a reasonable model for the nonlinear interaction between convection and the mode. We do not aim at a mathematically rigorous treatment of stochastic differential equations, but rather at getting a feel for the properties of the solution.

As a preparation, we first consider a simple damped oscillator, with no forcing.

i) Consider an oscillator with amplitude $A(t)$ which satisfies the differential equation

$$\frac{\mathrm{d}^2 A}{\mathrm{d}t^2} + 2\eta \frac{\mathrm{d}A}{\mathrm{d}t} + \omega_0^2 A = 0 \,, \tag{C.73}$$

with initial condition $A(0) = A_0$ and $\mathrm{d}A/\mathrm{d}t = 0$ at $t = 0$. Find the solution for $t > 0$ and show that for $\eta > 0$ it corresponds to a damped oscillation. You may assume that $|\eta| \ll \omega_0$.

ii) Find the Fourier transform of the solution from i), observed from $t = 0$ to $t \to \infty$, and the corresponding power spectrum [see also equation (2.38)]. This spectrum is known as a *Lorentz profile*.

We now consider an oscillator forced by a random function $f(t)$ and hence satisfying the equation

$$\frac{\mathrm{d}^2 A}{\mathrm{d}t^2} + 2\eta \frac{\mathrm{d}A}{\mathrm{d}t} + \omega_0^2 A = f(t) \,. \tag{C.74}$$

This equation is most easily dealt with in terms of its Fourier transform. We introduce the Fourier transforms $\tilde{A}(\omega)$ and $\tilde{f}(\omega)$ by

$$\tilde{A}(\omega) = \int A(t) e^{i\omega t} \mathrm{d}t \,, \qquad \tilde{f}(\omega) = \int f(t) e^{i\omega t} \mathrm{d}t \,, \tag{C.75}$$

where we do not attempt to specify the limits of integration precisely.

iii) Show, by suitable use of integration by parts and neglecting the resulting boundary terms, that $\tilde{A}$ satisfies

$$-\omega^2 \tilde{A} - 2i\eta\omega \tilde{A} + \omega_0^2 \tilde{A} = \tilde{f} \,. \tag{C.76}$$

iv) Show from equation (C.76) that the power spectrum of the oscillator is given by

$$P(\omega) = |\tilde{A}(\omega)|^2 = \frac{|\tilde{f}(\omega)|^2}{(\omega_0^2 - \omega^2)^2 + 4\eta^2\omega^2} \,. \tag{C.77}$$

Equation (C.77) describes the solution resulting from a particular realization of the forcing. It is more interesting to consider an average over several such realizations (obtained either by repeated observation of the same mode or by averaging data for several similar modes). Furthermore, since the damping rate is generally very small compared with the oscillation frequency, we are mainly interested in the behaviour close to $\omega = \omega_0$.

v) Show that for $|\omega - \omega_0| \ll \omega_0$ the average power of the oscillation, as a function of frequency, is given by

$$\langle P(\omega) \rangle \simeq \frac{1}{4\omega_0^2} \frac{\langle P_f(\omega) \rangle}{(\omega - \omega_0)^2 + \eta^2} , \qquad (C.78)$$

where $\langle P_f(\omega) \rangle$ is the average power of the forcing function.

Since $\langle P_f(\omega) \rangle$ is often a slowly varying function of frequency, the frequency-dependence of $\langle P(\omega) \rangle$ is dominated by the denominator in equation (C.78). This behaviour is exactly the same as the profile for the unforced damped oscillator in ii). Hence we obtain the remarkable result that the spectrum of a stochastically forced damped oscillator is Lorentzian, with a width determined by the linear damping rate η. Consequently, under the assumption of stochastic excitation one can make a meaningful comparison between computed damping rates and observed line widths. In the solar case rather detailed calculations by Balmforth (1992b), including a relatively sophisticated treatment of convection and radiation, have in fact resulted in good agreement with the observations. However, a more careful analysis of the statistical properties of the observed oscillations is required to confirm that this is indeed the correct model for the excitation of the solar modes. As discussed in Section 10.3, the observed amplitude distribution is in fact in accord with expectations. Based on such models of excitation, predictions have been carried out of amplitudes of similar oscillations in other stars, of obvious significance for attempts to detect such oscillations. In the longer run, we may hope to be able to probe properties of convection in different stars through observations of their oscillation amplitudes.

Finally it should be remarked that numerical simulations of such stochastically excited oscillators are both relatively straightforward and very instructive.

Problem 6.2:

The location of the instability strip in the HR diagram. We have found that instability requires coincidence of the He^+ ionization zone and the transition from adiabatic to nonadiabatic oscillations. Here we analyse this condition in more detail.

We consider the outer layers of a star. The mass of the layer is Δm, and its thickness is Δr; we assume that $\Delta m \ll M$ and $\Delta r \ll R$, where M and R are the mass and radius of the star. The pressure on the stellar surface is assumed to be zero.

i) Show from the equation of hydrostatic support that the pressure at the base of the layer is

$$p_1 = \frac{GM\Delta m}{4\pi R^4} , \qquad (C.79)$$

where G is the gravitational constant.

We assume that energy transport is through radiation, and that the opacity is given by the Kramers expression, which we write on the form

$$\kappa = \tilde{\kappa}_0 p T^{-4.5} , \qquad (C.80)$$

where $\tilde{\kappa}_0$ is a constant, and T is temperature.

ii) Show from the equations of hydrostatic support and radiative energy transport that the temperature T_1 at the base of the layer satisfies

$$p_1 \simeq K \left(\frac{M}{L}\right)^{1/2} T_1^{4.25} , \tag{C.81}$$

where L is the luminosity of the star and K is a constant.

In equation (10.40), it was argued that the transition from adiabaticity to nonadiabaticity occurs at a depth $(\Delta r)_{\rm TR}$ such that

$$\frac{\langle c_V T\rangle_{\rm TR}(\Delta m)_{\rm TR}}{L\Pi} \sim 1 , \tag{C.82}$$

where $\langle c_V T\rangle_{\rm TR}$ is an average of $c_V T$ over the region outside the transition point, $(\Delta m)_{\rm TR}$ is the mass of that region and Π is the pulsation period. The condition for instability is that $(T_1)_{\rm TR} \simeq T_{\rm ion}$, where $(T_1)_{\rm TR}$ is the temperature at the transition point and $T_{\rm ion} \simeq 4 \times 10^4\,{\rm K}$ is the temperature where He^+ ionizes.

We write the pulsation period as

$$\Pi = \Pi_0 \frac{R^{3/2}}{M^{1/2}} , \tag{C.83}$$

where Π_0 is a constant, and approximate $\langle c_V T\rangle_{\rm TR}$ by $c_V (T_1)_{\rm TR}$, where c_V is assumed to be constant.

iii) Show, using equations (C.79) and (C.81) – (C.83), that the instability condition leads to the following relation between L and R:

$$L \propto R^{5/3} . \tag{C.84}$$

iv) Show that equation (C.84) can be expressed as

$$L \propto T_{\rm eff}^{-\nu} , \tag{C.85}$$

and find the exponent ν.