Chapter 3

A little hydrodynamics

To provide a background for the presentation of the theory of stellar oscillations, this chapter briefly discusses some basic principles of hydrodynamics. A slightly more detailed description, but still essentially without derivations, was given by Cox (1980). In addition, any of the many detailed books on hydrodynamics (*e.g.* Batchelor 1967; Landau & Lifshitz 1959) can be consulted. Ledoux & Walraven (1958) give a very comprehensive introduction to hydrodynamics, with special emphasis on the application to stellar oscillations.

3.1 Basic equations of hydrodynamics

It is assumed that the gas can be treated as a continuum, so that its properties can be specified as functions of position $\mathbf{r}$ and time t. These properties include the local density $\rho(\mathbf{r}, t)$, the local pressure $p(\mathbf{r}, t)$ (and any other thermodynamic quantities that may be needed), as well as the local instantaneous velocity $\mathbf{v}(\mathbf{r}, t)$. Here $\mathbf{r}$ denotes the position vector to a given point in space, and the description therefore corresponds to what is seen by a stationary observer. This is known as the *Eulerian* description. In addition, it is often convenient to use the *Lagrangian* description, which is that of an observer who follows the motion of the gas. Here a given element of gas can be labelled, *e.g.* by its initial position $\mathbf{r}_0$, and its motion is specified by giving its position $\mathbf{r}(t, \mathbf{r}_0)$ as a function of time. Its velocity

$$\mathbf{v}(\mathbf{r}, t) = \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} \qquad \text{at fixed } \mathbf{r}_0 \tag{3.1}$$

is equivalent to the Eulerian velocity mentioned above.

The time derivative of a quantity ϕ, observed when following the motion is

$$\frac{\mathrm{d}\phi}{\mathrm{d}t} = \left(\frac{\partial \phi}{\partial t}\right)_{\mathbf{r}} + \nabla\phi \cdot \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} = \frac{\partial \phi}{\partial t} + \mathbf{v} \cdot \nabla\phi \; . \tag{3.2}$$

The time derivative $\mathrm{d}/\mathrm{d}t$ following the motion is also known as the material time derivative; in contrast $\partial/\partial t$ is the local time derivative (*i.e.*, the time derivative at a fixed point).

The properties of the gas are expressed as scalar and vector fields. Thus we need a little vector algebra; convenient summaries can be found, *e.g.* in books on electromagnetism (such as Jackson 1975; Reitz, Milford & Christy 1979). I shall assume the rules for manipulating gradients and divergences to be known. In addition, we need Gauss's theorem:

$$\int_{\partial V} \mathbf{a} \cdot \mathbf{n} \, \mathrm{d}A = \int_V \mathrm{div}\, \mathbf{a} \, \mathrm{d}V \; , \tag{3.3}$$

where V is a volume, with surface ∂V, $\mathbf{n}$ is the outward directed normal to ∂V, and $\mathbf{a}$ is any vector field. From this one also obtains

$$\int_{\partial V} \phi \mathbf{n} \, \mathrm{d}A = \int_V \nabla \phi \, \mathrm{d}V \tag{3.4}$$

for any scalar field ϕ.

3.1.1 The equation of continuity

The fact that mass is conserved can be expressed as

$$\frac{\partial \rho}{\partial t} + \mathrm{div}\,(\rho \mathbf{v}) = 0 \; , \tag{3.5}$$

where ρ is density. This is a typical conservation equation, balancing the rate of change of a quantity in a volume with the flux of the quantity into the volume. Had there been any sources of mass, they would have appeared on the right-hand side. By using the relation (3.2), equation (3.5) may also be written

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} + \rho \, \mathrm{div}\, \mathbf{v} = 0 \; , \tag{3.6}$$

giving the rate of change of density following the motion. Note that $\rho = 1/V$, where V is the volume of unit mass; thus an alternative formulation is

$$\frac{1}{V} \frac{\mathrm{d}V}{\mathrm{d}t} = \mathrm{div}\, \mathbf{v} \; . \tag{3.7}$$

Hence $\mathrm{div}\, \mathbf{v}$ is the rate of expansion of a given volume of gas, when following the motion.

3.1.2 Equations of motion

Under solar or stellar conditions one can generally ignore the internal friction (or *viscosity*) in the gas. The forces on a volume of gas therefore consist of

i) Surface forces, *i.e.*, the pressure on the surface of the volume

ii) Body forces.

Thus the equations of motion can be written

$$\rho \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = -\nabla p + \rho \, \mathbf{f} \; , \tag{3.8}$$

where $\mathbf{f}$ is the body force per unit mass which has yet to be specified. The pressure p is defined such that the force on a surface element $\mathrm{d}A$ with outward normal $\mathbf{n}$ is $-p \, \mathbf{n} \, \mathrm{d}A$. This may be identified with the ordinary thermodynamic pressure.

By using equation (3.2), we may also write equation (3.8) as

$$\rho \frac{\partial \mathbf{v}}{\partial t} + \rho \mathbf{v} \cdot \nabla \mathbf{v} = - \nabla p + \rho \mathbf{f} \; . \tag{3.9}$$

Among the possible body forces I consider only gravity. Thus in particular I neglect effects of magnetic fields, which might otherwise provide a body force on the gas. The force per unit mass from gravity is the gravitational acceleration $\mathbf{g}$, which can be written as the gradient of the gravitational potential Φ:

$$\mathbf{g} = -\nabla \Phi \; , \tag{3.10}$$

where Φ satisfies Poisson's equation

$$\nabla^2 \Phi = 4\pi G \rho \; . \tag{3.11}$$

It is often convenient to use also the integral solution to Poisson's equation

$$\Phi(\mathbf{r}, t) = -G \int_V \frac{\rho(\mathbf{r}', t) \mathrm{d}V}{|\mathbf{r} - \mathbf{r}'|} \; . \tag{3.12}$$

3.1.3 Energy equation

To complete the equations we need a relation between p and ρ. This must take the form of a thermodynamic relation. Specifically the first law of thermodynamics,

$$\frac{\mathrm{d}q}{\mathrm{d}t} = \frac{\mathrm{d}E}{\mathrm{d}t} + p \frac{\mathrm{d}V}{\mathrm{d}t} \; , \tag{3.13}$$

must be satisfied; here $\mathrm{d}q/\mathrm{d}t$ is the rate of heat loss or gain, and E the internal energy, per unit mass. As before $V = 1/\rho$ is specific volume. Thus equation (3.13) expresses the fact that the heat gain goes partly to change the internal energy, partly into work expanding or compressing the gas. Alternative formulations of equation (3.13), using the equation of continuity, are

$$\frac{\mathrm{d}q}{\mathrm{d}t} = \frac{\mathrm{d}E}{\mathrm{d}t} - \frac{p}{\rho^2} \frac{\mathrm{d}\rho}{\mathrm{d}t} = \frac{\mathrm{d}E}{\mathrm{d}t} + \frac{p}{\rho} \operatorname{div} \mathbf{v} \; . \tag{3.14}$$

By using thermodynamic identities the energy equation can be expressed in terms of other, and more convenient, variables.

$$\frac{\mathrm{d}q}{\mathrm{d}t} = \frac{1}{\rho(\Gamma_3 - 1)} \left(\frac{\mathrm{d}p}{\mathrm{d}t} - \frac{\Gamma_1 p}{\rho} \frac{\mathrm{d}\rho}{\mathrm{d}t} \right) \tag{3.15}$$

$$= c_p \left(\frac{\mathrm{d}T}{\mathrm{d}t} - \frac{\Gamma_2 - 1}{\Gamma_2} \frac{T}{p} \frac{\mathrm{d}p}{\mathrm{d}t} \right) \tag{3.16}$$

$$= c_V \left[\frac{\mathrm{d}T}{\mathrm{d}t} - (\Gamma_3 - 1) \frac{T}{\rho} \frac{\mathrm{d}\rho}{\mathrm{d}t} \right] \; . \tag{3.17}$$

Here c_p and c_V are the specific heat per unit mass at constant pressure and volume, and the adiabatic exponents are defined by

$$\Gamma_1 = \left(\frac{\partial \ln p}{\partial \ln \rho} \right)_{\mathrm{ad}} , \quad \frac{\Gamma_2 - 1}{\Gamma_2} = \left(\frac{\partial \ln T}{\partial \ln p} \right)_{\mathrm{ad}} , \quad \Gamma_3 - 1 = \left(\frac{\partial \ln T}{\partial \ln \rho} \right)_{\mathrm{ad}} \; . \tag{3.18}$$

These relations are discussed in more detail in, *e.g.*, Cox & Giuli (1968).

It is evident that the relation between p, ρ and T, as well as the Γ_i's, depend on the thermodynamic state and composition of the gas. Indeed, as will be discussed below, the dependence of Γ_1 on the properties of the gas forms the basis for using observed solar oscillation frequencies to probe the details of the statistical mechanics of partially ionized gases and to infer the helium abundance of the solar convective envelope. However, in many cases one may as a first approximation regard the gas as fully ionized and neglect effects of degeneracy and radiation pressure. Then the equation of state is simply

$$p = \frac{k_{\rm B} \rho T}{\mu m_{\rm u}} , \tag{3.19}$$

where $k_{\rm B}$ is Boltzmann's constant, $m_{\rm u}$ is the atomic mass unit and μ is the mean molecular weight. Also

$$\Gamma_1 = \Gamma_2 = \Gamma_3 = 5/3 . \tag{3.20}$$

I note that radiation pressure decreases Γ_1 below this value; this effect becomes noticeable in stars whose mass exceeds a few solar masses. Thus Otzen Petersen (1975) showed that radiation pressure caused a systematic increase of the pulsation constant (*cf.* eq. 2.20) with increasing luminosity along the Cepheid instability strip.

We need to consider the heat gain in more detail. Specifically, it can be written as

$$\rho \frac{{\rm d}q}{{\rm d}t} = \rho \, \epsilon - {\rm div} \, \mathbf{F} ; \tag{3.21}$$

here ϵ is the rate of energy generation per unit mass (*e.g.* from nuclear reactions), and $\mathbf{F}$ is the flux of energy. In general, radiation is the only significant contributor to the energy flux; in particular, molecular conduction is almost always negligible.

In convection zones turbulent gas motion provides a very efficient transport of energy. Ideally the entire hydrodynamical system, including convection, must be described as a whole. In this case only the radiative flux would be included in equation (3.21). However, under most circumstances the resulting equations are too complex to be handled analytically or numerically. Thus it is customary to separate out the convective motion, by performing averages of the equations over length scales that are large compared with the convective motion, but small compared with other scales of interest. In this case the convective flux appears as an additional contribution in equation (3.21). The convective flux must then be determined, from the other quantities characterizing the system, by considering the equations for the turbulent motion. A familiar example of this (which is also characteristic of the lack of sophistication in current treatments of convection) is the mixing-length theory.

The incorporation of convection in the hydrodynamical equations was discussed in some detail by Unno *et al.* (1989). However, it is fair to say that this is currently one of the principal uncertainties in stellar hydrodynamics.

The general calculation of the radiative flux is also non-trivial. In stellar atmospheres the full radiative transfer problem, as known from the theory of the structure of stellar atmospheres, must be solved in combination with the hydrodynamic equations. This is another active area of research, and the subject of a major monograph (Mihalas & Mihalas 1984). In stellar interiors the diffusion approximation is adequate, and the radiative flux is given by

$$\mathbf{F} = -\frac{4\pi}{3\kappa\rho} \nabla B = -\frac{4 a \tilde{c} T^3}{3\kappa\rho} \nabla T , \tag{3.22}$$

where $B = (a\tilde{c}/4\pi)T^4$ is the integrated Planck function, κ is the opacity, $\tilde{c}$ is the speed of light and a is the radiation density constant; this provides a relation between the state of the gas and the radiative flux, which is analogous to a simple conduction equation.

When the mean free path of a photon is very large, one can neglect the contribution from absorption to the heating of the gas. Then we have that

$$\operatorname{div} \mathbf{F} = 4\pi\rho\kappa_{\rm a} B \; , \tag{3.23}$$

where $\kappa_{\rm a}$ is the opacity arising from absorption; this is the so-called *Newton's law of cooling.* Finally, one can generalize the Eddington approximation, which may be known from the theory of static stellar atmospheres, to the three-dimensional case (see Unno & Spiegel 1966), to obtain

$$\operatorname{div} \mathbf{F} = -4\pi\rho\kappa_{\rm a}(J - B) \; , \tag{3.24}$$

$$\mathbf{F} = -\frac{4\pi}{3(\kappa_{\rm a} + \kappa_{\rm s})\rho} \nabla J \; , \tag{3.25}$$

where $\kappa_{\rm s}$ is the scattering opacity and J is the mean intensity. As shown by Unno & Spiegel the Eddington approximation tends to the diffusion approximation when $\kappa_{\rm a}\rho \to \infty$. Furthermore, it has the correct limit in the optically thin case.

Here I have implicitly assumed that the scattering and absorption coefficients are independent of the frequency of radiation. In the diffusion approximation, the generalization to frequency-dependence leads to the introduction of the Rosseland mean opacity. In the optically thin case, one must in general take into account the details of the distribution of intensity with frequency; thus, in equations (3.23) – (3.25) the absorption and scattering coefficients must be thought of as suitable averages, whereas $\mathbf{F}$ and J are frequency-integrated quantities.

3.1.4 The adiabatic approximation

For the purpose of calculating stellar oscillation frequencies, the complications of the energy equation can be avoided to a high degree of precision, by neglecting the heating term in the energy equation. To see that this is justified, consider the energy equation on the form, using equation (3.22)

$$\frac{{\rm d}T}{{\rm d}t} - \frac{\Gamma_2 - 1}{\Gamma_2}\frac{T}{p}\frac{{\rm d}p}{{\rm d}t} = \frac{1}{c_p}\left[\epsilon + \frac{1}{\rho}\operatorname{div}\left(\frac{4a\tilde{c}T^3}{3\kappa\rho}\nabla T\right)\right] \; . \tag{3.26}$$

Here the term in the temperature gradient can be estimated as

$$\frac{1}{\rho c_p}\operatorname{div}\left(\frac{4a\tilde{c}T^3}{3\kappa\rho}\nabla T\right) \sim \frac{4a\tilde{c}T^4}{3\kappa\rho^2 c_p \mathcal{L}^2} = \frac{T}{\tau_F} \; , \tag{3.27}$$

where $\mathcal{L}$ is a characteristic length scale, and τ_F is a characteristic time scale for radiation,

$$\tau_F = \frac{3\kappa\rho^2 c_p \mathcal{L}^2}{4a\tilde{c}T^3} \simeq 10^{12}\frac{\kappa\rho^2\mathcal{L}^2}{T^3} \; , \qquad \text{in cgs units} \; . \tag{3.28}$$

Typical values for the entire Sun are $\kappa = 1$, $\rho = 1$, $T = 10^6$, $\mathcal{L} = 10^{10}$, and hence $\tau_F \sim 10^7$ years. This corresponds to the Kelvin-Helmholtz time for the star. For the solar convection zone the corresponding values are $\kappa = 100$, $\rho = 10^{-5}$, $T = 10^4$, $\mathcal{L} = 10^9$, and hence

$\tau_F \sim 10^3$ years. In the outer parts of the star the term in ϵ vanishes, whereas in the core it corresponds to a characteristic time $\tau_\epsilon \sim c_p T/\epsilon$ which is again of the order of the Kelvin-Helmholtz time. T/τ_F or T/τ_ϵ must be compared with the time derivative of T in equation (3.26), which can be estimated as $T/$(period of oscillation). Typical periods are of the order of minutes to hours, and hence the heating term in equation (3.26) is generally very small compared with the time-derivative terms. Near the surface, on the other hand, the density, and hence the radiative time scale, is low, and the full energy equation must be taken into account.

Where the heating can be neglected, the motion occurs *adiabatically*. Then p and ρ are related by

$$\frac{\mathrm{d}p}{\mathrm{d}t} = \frac{\Gamma_1 p}{\rho} \frac{\mathrm{d}\rho}{\mathrm{d}t} . \tag{3.29}$$

This equation, together with the continuity equation (3.5), the equations of motion (3.9) and Poisson's equation (3.11), form the complete set of equations for adiabatic motion. Most of our subsequent work is based on these equations.

3.2 Equilibrium states and perturbation analysis

A general hydrodynamical description of a star, using the equations presented in the preceding section, is far too complex to handle, even numerically on the largest existing computers. To put this in perspective, it may be mentioned that Å. Nordlund and R. Stein (*e.g.* Nordlund & Stein 1989; Stein, Nordlund & Kuhn 1989), by stretching the capabilities of existing computers to the limits, have been able to follow numerically the development of a very small region near the solar surface for a few hours. Even though this is a tremendous achievement, which will be of great value to our understanding of solar convection and solar oscillations, it clearly demonstrates the impracticality of a direct numerical solution for, say, general oscillations involving the entire Sun. Furthermore, even to the extent that such a solution were possible, the results would in general be so complicated that a simplified analysis is needed to understand them. Fortunately, in the case of stellar oscillations, considerable simplifications are possible. The observed solar oscillations have very small amplitudes compared with the characteristic scales of the Sun, and so it can be treated as a small perturbation around a static equilibrium state. Even in "classical" pulsating stars, where the surface amplitudes are large, most of the energy in the motion is in regions where the amplitudes are relatively small; thus many of the properties of these oscillations, including their periods, can be understood in terms of small-perturbation theory. In this section I discuss the general equations for such small perturbations.

3.2.1 The equilibrium structure

The equilibrium structure is assumed to be static, so that all time derivatives can be neglected. In addition, I assume that there are no velocities. Then the continuity equation, (3.5), is trivially satisfied. The equations of motion (3.9) reduce to the equation of hydrostatic support,

$$\nabla p_0 = \rho_0 \mathbf{g}_0 = -\rho_0 \nabla \Phi_0 , \tag{3.30}$$

where I have denoted equilibrium quantities with the subscript "0". Poisson's equation (3.11) is unchanged,

$$\nabla^2 \Phi_0 = 4\pi G \rho_0 . \tag{3.31}$$

Finally the energy equation (3.21) is

$$0 = \frac{\mathrm{d}q}{\mathrm{d}t} = \epsilon_0 - \frac{1}{\rho_0} \mathrm{div}\, \mathbf{F}_0 \; . \tag{3.32}$$

It might be noted that one often considers equilibrium structures that change on long time scales. Here hydrostatic equilibrium is enforced (departures from hydrostatic equilibrium result in motion on essentially the free-fall time scale for the star, of the order of hours). However, it is not assumed that there is no heating, so that the general energy equation (3.21) is used. Such a star is said to be in hydrostatic, but not in thermal, equilibrium. Typical examples are stars where nuclear burning does not supply the main source of energy, as during the pre-main-sequence contraction, or after hydrogen exhaustion in the core. Even during normal main sequence evolution the heating term provides a small contribution to the energy, which is normally taken into account in calculations of stellar evolution. However, we need not consider this further here.

For the present purpose the most important example of equilibrium is clearly a spherically symmetric state, where the structure depends only on the distance r to the centre. Here $\mathbf{g}_0 = -g_0 \mathbf{a}_r$, where $\mathbf{a}_r$ is a unit vector directed radially outward, and equation (3.30) becomes

$$\frac{\mathrm{d}p_0}{\mathrm{d}r} = -g_0 \rho_0 \; . \tag{3.33}$$

Also, Poisson's equation may be integrated once, to yield

$$g_0 = \frac{G}{r^2} \int_0^r 4\pi \rho_0 r'^2 \, \mathrm{d}r' = \frac{G\, m_0}{r^2} \; , \tag{3.34}$$

where $m_0(r)$ is the mass in the sphere interior to r. The flux is directed radially outward, $\mathbf{F} = F_{r,0} \mathbf{a}_r$, so that the energy equation gives

$$\rho_0 \epsilon_0 = \frac{1}{r^2} \frac{\mathrm{d}}{\mathrm{d}r} \left(r^2 \, F_{r,0} \right) = \frac{1}{4\pi r^2} \frac{\mathrm{d}L_0}{\mathrm{d}r} \; ,$$

where $L_0 = 4\pi r^2 \, F_{r,0}$ is the total flow of energy through the sphere with radius r; hence

$$\frac{\mathrm{d}L_0}{\mathrm{d}r} = 4\pi r^2 \rho_0 \epsilon_0 \; . \tag{3.35}$$

Finally the diffusion expression (3.22) for the flux may be written

$$\frac{\mathrm{d}T_0}{\mathrm{d}r} = - \frac{3\kappa_0 \rho_0}{16\pi r^2 a \tilde{c} T_0^3} L_0 \; . \tag{3.36}$$

Equations (3.33) – (3.36) are clearly the familiar equations for stellar structure.

3.2.2 Perturbation analysis

We consider small perturbations around the equilibrium state. Thus, *e.g.*, the pressure is written as

$$p(\mathbf{r}, t) = p_0(\mathbf{r}) + p'(\mathbf{r}, t) \; , \tag{3.37}$$

where p' is a small perturbation; this is the so-called *Eulerian* perturbation, *i.e.*, the perturbation at a given point. The equations are then linearized in the perturbations, by expanding them in the perturbations retaining only terms that do not contain products of the perturbations.

Just as in the general case it is convenient to use also a description involving a reference frame following the motion; the perturbation in this frame is called the *Lagrangian* perturbation. If an element of gas is moved from $\mathbf{r}_0$ to $\mathbf{r}_0 + \boldsymbol{\delta}\mathbf{r}$ due to the perturbation, the Lagrangian perturbation to pressure may be calculated as

$$\begin{aligned} \delta p(\mathbf{r}) &= p(\mathbf{r}_0 + \boldsymbol{\delta}\mathbf{r}) - p_0(\mathbf{r}_0) = p(\mathbf{r}_0) + \boldsymbol{\delta}\mathbf{r} \cdot \nabla p_0 - p_0(\mathbf{r}_0) \\ &= p'(\mathbf{r}_0) + \boldsymbol{\delta}\mathbf{r} \cdot \nabla p_0 \ . \end{aligned} \tag{3.38}$$

Equation (3.38) is of course completely equivalent to the relation (3.2) between the local and the material time derivative. Note also that the velocity is given by the time derivative of the displacement $\boldsymbol{\delta}\mathbf{r}$,

$$\mathbf{v} = \frac{\partial \boldsymbol{\delta}\mathbf{r}}{\partial t} \ . \tag{3.39}$$

Equations for the perturbations are obtained by inserting expressions like (3.37) in the full equations, subtracting equilibrium equations and neglecting quantities of order higher than one in p', ρ', $\mathbf{v}$, *etc.* For the continuity equation the result is

$$\frac{\partial \rho'}{\partial t} + \mathrm{div}\,(\rho_0 \mathbf{v}) = 0 \ , \tag{3.40}$$

or, by using equation (3.39) and integrating with respect to time

$$\rho' + \mathrm{div}\,(\rho_0 \boldsymbol{\delta}\mathbf{r}) = 0 \ . \tag{3.41}$$

Note that this equation may also, by using the analogue to equation (3.38), be written as

$$\delta\rho + \rho_0 \mathrm{div}\,(\boldsymbol{\delta}\mathbf{r}) = 0 \ , \tag{3.42}$$

which corresponds to equation (3.6).

The equations of motion become

$$\rho_0 \frac{\partial^2 \boldsymbol{\delta}\mathbf{r}}{\partial t^2} = \rho_0 \frac{\partial \mathbf{v}}{\partial t} = -\nabla p' + \rho_0 \mathbf{g}' + \rho' \mathbf{g}_0 \ , \tag{3.43}$$

where, obviously, $\mathbf{g}' = -\nabla \Phi'$. Also, the perturbation Φ' in the gravitational potential satisfies the perturbed Poisson's equation

$$\nabla^2 \Phi' = 4\pi G \rho' \ , \tag{3.44}$$

with the solution, equivalent to equation (3.12)

$$\Phi' = -G \int_V \frac{\rho'(\mathbf{r}', t)}{|\mathbf{r} - \mathbf{r}'|} \mathrm{d}V \ . \tag{3.45}$$

The energy equation requires a little thought. We need to calculate, *e.g.*,

$$\frac{\mathrm{d}p}{\mathrm{d}t} = \frac{\partial p}{\partial t} + \mathbf{v} \cdot \nabla p = \frac{\partial p'}{\partial t} + \mathbf{v} \cdot \nabla p_0 = \frac{\partial p'}{\partial t} + \frac{\partial \boldsymbol{\delta}\mathbf{r}}{\partial t} \cdot \nabla p_0 = \frac{\partial}{\partial t}(\delta p) \ , \tag{3.46}$$

to first order in the perturbations. Note that to this order there is no difference between the local and the material time derivative of the *perturbations*. Thus we have for the energy equation, from *e.g.* equation (3.15),

$$\frac{\partial \delta q}{\partial t} = \frac{1}{\rho_0(\Gamma_{3,0} - 1)} \left(\frac{\partial \delta p}{\partial t} - \frac{\Gamma_{1,0} p_0}{\rho_0} \frac{\partial \delta \rho}{\partial t} \right) . \tag{3.47}$$

This equation is most simply expressed in Lagrangian perturbations, but it may be transformed into Eulerian perturbations by using equation (3.38). From equation (3.21) the perturbation to the heating rate is given by

$$\rho_0 \frac{\partial \delta q}{\partial t} = \delta(\rho\epsilon - \operatorname{div} \mathbf{F}) = (\rho\epsilon - \operatorname{div} \mathbf{F})' , \tag{3.48}$$

if equation (3.32) is used. Finally it is straightforward to obtain the perturbation to the radiative flux, in the diffusion approximation, from equation (3.22).

For adiabatic motion we neglect the heating term and obtain

$$\frac{\partial \delta p}{\partial t} - \frac{\Gamma_{1,0} p_0}{\rho_0} \frac{\partial \delta \rho}{\partial t} = 0 ,$$

or, by integrating over time

$$\delta p = \frac{\Gamma_{1,0} p_0}{\rho_0} \delta \rho , \tag{3.49}$$

or, on Eulerian form

$$p' + \boldsymbol{\delta}\mathbf{r} \cdot \nabla p_0 = \frac{\Gamma_{1,0} p_0}{\rho_0} (\rho' + \boldsymbol{\delta}\mathbf{r} \cdot \nabla \rho_0) . \tag{3.50}$$

3.3 Simple waves

It is instructive to consider simple examples of wave motion. This provides an introduction to the techniques needed to handle the perturbations. In addition, general stellar oscillations can in many cases be approximated by simple waves, which therefore give physical insight into the behaviour of the oscillations.

3.3.1 Acoustic waves

As the simplest possible equilibrium situation, we may consider the spatially homogeneous case. Here all derivatives of equilibrium quantities vanish. According to equation (3.30) gravity must then be negligible. Such a situation clearly cannot be realized exactly. However, if the equilibrium structure varies slowly compared with the oscillations, this may be a reasonable approximation. I also neglect the perturbation to the gravitational potential; for rapidly varying perturbations regions with positive and negative ρ' nearly cancel in equation (3.45), and hence Φ' is small. Finally, I assume the adiabatic approximation (3.49).

The equations of motion (3.43) give

$$\rho_0 \frac{\partial^2 \boldsymbol{\delta}\mathbf{r}}{\partial t^2} = -\nabla p' ,$$

or, by taking the divergence

$$\rho_0 \frac{\partial^2}{\partial t^2}(\operatorname{div} \boldsymbol{\delta}\mathbf{r}) = -\nabla^2 p' \; .$$

However, $\operatorname{div} \boldsymbol{\delta}\mathbf{r}$ can be eliminated by using the continuity equation (3.41), and p' can be expressed in terms of ρ' from the adiabatic relation. The result is

$$\frac{\partial^2 \rho'}{\partial t^2} = \frac{\Gamma_{1,0} p_0}{\rho_0} \nabla^2 \rho' = c_0^2 \nabla^2 \rho' \; , \tag{3.51}$$

where

$$c_0^2 \equiv \frac{\Gamma_{1,0} p_0}{\rho_0} \tag{3.52}$$

has the dimension of a squared velocity. This equation has the form of the wave equation. Thus it has solutions in the form of plane waves

$$\rho' = a \exp[i(\mathbf{k} \cdot \mathbf{r} - \omega\, t)] \; . \tag{3.53}$$

(As discussed in more detail in Chapter 4 it is convenient to write the solution in complex form; the physically realistic solution is obtained by taking the real part of the complex solution.) By substituting equation (3.53) into (3.51) we obtain

$$-\omega^2 \rho' = c_0^2 \operatorname{div}(i \mathbf{k} \rho') = -c_0^2 |\mathbf{k}|^2 \rho' \; . \tag{3.54}$$

Thus this is a solution, provided ω satisfies the *dispersion relation*

$$\omega^2 = c_0^2 \, |\mathbf{k}|^2 \; . \tag{3.55}$$

The waves are plane sound waves, and equation (3.55) is the dispersion relation for such waves. The adiabatic sound speed c_0 is the speed of propagation of the waves. I note that when the ideal gas law, equation (3.19), is satisfied, the sound speed is given by

$$c_0^2 = \frac{\Gamma_{1,0} k_{\mathrm{B}} T_0}{\mu m_{\mathrm{u}}} \; . \tag{3.56}$$

Thus c_0 is essentially determined by T_0/μ.

With a suitable choice of phases the real solution can be written as

$$\rho' = a \cos(\mathbf{k} \cdot \mathbf{r} - \omega t) \; , \tag{3.57}$$

$$p' = c_0^2 a \cos(\mathbf{k} \cdot \mathbf{r} - \omega t) \; , \tag{3.58}$$

$$\boldsymbol{\delta}\mathbf{r} = \frac{c_0^2}{\rho_0 \omega^2} a \, \cos(\mathbf{k} \cdot \mathbf{r} - \omega t + \frac{\pi}{2})\, \mathbf{k} \; . \tag{3.59}$$

Thus the displacement $\boldsymbol{\delta}\mathbf{r}$, and hence the velocity $\mathbf{v}$, is in the direction of the wave vector $\mathbf{k}$.

3.3.2 Internal gravity waves

As a slightly more complicated case, we consider a layer of gas stratified under gravity. Thus here there is a pressure gradient, determined by equation (3.33). However, I assume that the equilibrium quantities vary so slowly that their gradients can be neglected compared with gradients of perturbations. Also, as before, I neglect the perturbation to the gravitational potential. Clearly one solution must be the adiabatic sound waves considered above. However, here we seek other solutions in the form of waves with much lower frequencies.

It is possible to derive an approximate wave equation under these circumstances (*cf.* Section 7.5). However, to simplify the analysis I assume a solution in the form of a plane wave from the outset. Thus I take all perturbation variables to vary as

$$\exp[i(\mathbf{k}\cdot\mathbf{r}-\omega t)] \; . \tag{3.60}$$

Because of the presence of gravity there is a preferred direction in the problem. I choose a vertical coordinate r directed upward, so that $\mathbf{g}_0 = -g_0\mathbf{a}_r$, and

$$\nabla p_0 = \frac{\mathrm{d}p_0}{\mathrm{d}r}\mathbf{a}_r \; , \qquad \nabla \rho_0 = \frac{\mathrm{d}\rho_0}{\mathrm{d}r}\mathbf{a}_r \; . \tag{3.61}$$

Also, I separate the displacement $\boldsymbol{\delta}\mathbf{r}$ and the wave vector $\mathbf{k}$ into radial and horizontal components,

$$\boldsymbol{\delta}\mathbf{r} = \xi_r\mathbf{a}_r + \boldsymbol{\xi}_\mathrm{h} \; , \tag{3.62}$$

$$\mathbf{k} = k_r\mathbf{a}_r + \mathbf{k}_\mathrm{h} \; . \tag{3.63}$$

The radial and horizontal components of the equations (3.43) are

$$-\rho_0\omega^2\xi_r = -ik_r p' - \rho' g_0 \; , \tag{3.64}$$

$$-\rho_0\omega^2\boldsymbol{\xi}_\mathrm{h} = -i\,\mathbf{k}_\mathrm{h} p' \; , \tag{3.65}$$

and the continuity equation, (3.41), can be written

$$\rho' + \rho_0 i k_r \xi_r + \rho_0 i\,\mathbf{k}_\mathrm{h}\cdot\boldsymbol{\xi}_\mathrm{h} = 0 \; . \tag{3.66}$$

From equations (3.65) and (3.66) we find the pressure perturbation

$$p' = \frac{\omega^2}{k_\mathrm{h}^2}(\rho' + ik_r\rho_0\xi_r) \; . \tag{3.67}$$

This may be used in equation (3.64), to obtain

$$-\rho_0\omega^2\xi_r = -i\frac{k_r}{k_\mathrm{h}^2}\omega^2\rho' + \omega^2\rho_0\frac{k_r^2}{k_\mathrm{h}^2}\xi_r - \rho' g_0 \; . \tag{3.68}$$

For very small frequency the first term in ρ' can be neglected compared with the second, yielding

$$\rho_0\omega^2\left(1+\frac{k_r^2}{k_\mathrm{h}^2}\right)\xi_r = \rho' g_0 \; . \tag{3.69}$$

Notice that this equation has a fairly simple physical meaning. Buoyancy acting on the density perturbation provides a vertical force $\rho' g_0$ per unit volume that drives the motion. The left-hand side gives the vertical acceleration times the mass ρ_0 per unit volume; however, it is modified by the term in the wave numbers. This arises from the pressure perturbation; in order to move vertically, a blob of gas must displace matter horizontally, and this increases its effective inertia. This effect is stronger the longer the horizontal wavelength of the perturbation, and hence the smaller its horizontal wave number.

The adiabatic relation (3.50) gives

$$\rho' = c_0^{-2} p' + \rho_0 \boldsymbol{\delta} \mathbf{r} \cdot \left(\frac{1}{p_0 \Gamma_{1,0}} \nabla p_0 - \frac{1}{\rho_0} \nabla \rho_0 \right) . \tag{3.70}$$

However, we may estimate the importance of the term in p' by noting that, according to equation (3.67),

$$\frac{c_0^{-2} p'}{\rho'} \simeq \frac{\omega^2}{c_0^2 k_{\rm h}^2} . \tag{3.71}$$

Here the denominator on the right-hand side is the sound-wave frequency corresponding to the horizontal wave number $k_{\rm h}$ (*cf.* eq. 3.55); since we are specifically interested in oscillations with frequencies far smaller than the frequencies of sound waves, this term can be neglected. Physically, the neglect of the pressure perturbation essentially corresponds to assuming that the perturbation is always in hydrostatic equilibrium; this might be compared with the conventional discussion of convective stability in terms of displaced blobs of fluid, where pressure balance is also assumed. Inserting the expression for ρ' resulting from equation (3.70), when p' is neglected, in equation (3.69) finally yields

$$\omega^2 \left(1 + \frac{k_r^2}{k_{\rm h}^2} \right) \xi_r = N^2 \xi_r , \tag{3.72}$$

where

$$N^2 = g_0 \left(\frac{1}{\Gamma_{1,0}} \frac{{\rm d} \ln p_0}{{\rm d} r} - \frac{{\rm d} \ln \rho_0}{{\rm d} r} \right) \tag{3.73}$$

is the square of the *buoyancy or Brunt-Väisälä frequency* N.

The physical significance of N follows from the 'blob' argument for convective stability (*e.g.* Christensen-Dalsgaard 1993a; see also Cox 1980, Section 17.2): if a fluid element is displaced upwards adiabatically, its behaviour depends on whether the density of the element is higher or smaller than its new surroundings. When $N^2 > 0$ the element is heavier than the displaced fluid, and buoyancy forces it back towards the original position; thus in this case the element executes an oscillation around the equilibrium position. On the other hand, if $N^2 < 0$ the element is lighter than the displaced fluid and buoyancy acts to enhance the motion, forcing the element away from equilibrium; this corresponds to convective instability.

From equation (3.72) we obtain the dispersion relation

$$\omega^2 = \frac{N^2}{1 + k_r^2 / k_{\rm h}^2} . \tag{3.74}$$

When $N^2 > 0$ the motion is oscillatory. Then N is the frequency in the limit of infinite $k_{\rm h}$, *i.e.*, for infinitely small horizontal wavelength. This corresponds to oscillations of fluid elements in the form of slender "needles". For greater horizontal wavelength the horizontal motion increases the inertia, as discussed above, and hence decreases the frequency. These waves are known as internal *gravity waves* (not to be confused with the gravitational waves in general relativity).

The condition that $N^2 > 0$ can also be written as

$$\frac{{\rm d} \ln \rho_0}{{\rm d} \ln p_0} > \frac{1}{\Gamma_{1,0}} ; \tag{3.75}$$

when it is not satisfied, ω is imaginary, and the motion grows exponentially with time. This is the linear case of convective instability. In general the motion grows until it breaks down into turbulence due to nonlinear effects. Thus gravity waves cannot propagate in convective regions. I return to this when discussing the asymptotic theory of stellar oscillations.

The condition (3.75) is the proper criterion for convective stability; it is normally known as *the Ledoux condition*. The more usual condition, in terms of p and T, *viz*

$$\frac{\mathrm{d}\ln T_0}{\mathrm{d}\ln \rho_0} < \nabla_{\mathrm{ad}} = \frac{\Gamma_{2,0} - 1}{\Gamma_{2,0}} , \tag{3.76}$$

can be obtained from equation (3.74) by using thermodynamic identities, when the chemical composition is homogeneous. Equation (3.76) is known as the *Schwarzschild criterion*. When there are gradients in the chemical composition, the two conditions are *not* equivalent. Nonetheless, the Schwarzschild criterion is most often used in calculations of stellar evolution, due to computational convenience.

3.3.3 Surface gravity waves

In addition to the internal gravity waves described above, there is a distinct, and more familiar, type of gravity waves, known, *e.g.* from the Bay of Aarhus. These are waves at a discontinuity in density.

We consider a liquid at constant density ρ_0, with a free surface. Thus the pressure on the surface is assumed to be constant. The layer is infinitely deep. I assume that the liquid is incompressible, so that ρ_0 is constant and the density perturbation $\rho' = 0$. From the equation of continuity we therefore get

$$\mathrm{div}\,\mathbf{v} = 0 . \tag{3.77}$$

Gravity $\mathbf{g}$ is assumed to be uniform, and directed vertically downwards. Since the density perturbation is zero, so is the perturbation to the gravitational potential.

In the interior of the liquid the equations of motion reduce to

$$\rho_0 \frac{\partial \mathbf{v}}{\partial t} = -\nabla p' . \tag{3.78}$$

The divergence of this equation gives

$$\nabla^2 p' = 0 . \tag{3.79}$$

We introduce a horizontal coordinate x, and a vertical coordinate z increasing downward, with $z = 0$ at the free surface. We now seek a solution in the form of a wave propagating along the surface, in the x-direction. Here p' has the form

$$p'(x, z, t) = f(z) \cos(k_{\mathrm{h}} x - \omega t) , \tag{3.80}$$

where f is a function yet to be determined. By substituting equation (3.80) into equation (3.79) we obtain

$$\frac{\mathrm{d}^2 f}{\mathrm{d}z^2} = k_{\mathrm{h}}^2 f ,$$

or

$$f(z) = a \exp(-k_{\mathrm{h}} z) + b \exp(k_{\mathrm{h}} z) . \tag{3.81}$$

As the layer is assumed to be infinitely deep, b must be zero.

We must now consider the boundary condition at the free surface. Here the pressure is constant, and therefore the Lagrangian pressure perturbation vanishes (the pressure is constant on the perturbed surface), *i.e.*,

$$0 = \delta p = p' + \boldsymbol{\delta}\mathbf{r} \cdot \nabla p_0 = p' + \xi_z \rho_0 g_0 \qquad \text{at } z = 0 \; , \tag{3.82}$$

where ξ_z is the z-component of the displacement. This is obtained from the vertical component of equation (3.78), for the solution in equation (3.81) with $b = 0$, as

$$\xi_z = -\frac{k_\mathrm{h}}{\rho_0 \omega^2} \, p' \; . \tag{3.83}$$

Thus equation (3.82) reduces to

$$0 = \left(1 - \frac{g_0 k_\mathrm{h}}{\omega^2}\right) p' \; ,$$

and hence the dispersion relation for the surface waves is

$$\omega^2 = g_0 k_\mathrm{h} \; . \tag{3.84}$$

The frequencies of the surface gravity waves depend only on their wavelength and on gravity, but not on the internal structure of the layer, in particular the density. In this they resemble a pendulum, whose frequency is also independent of its constitution. Indeed, the frequency of a wave with wave number k_h, and wavelength λ, is the same as the frequency of a mathematical pendulum with length

$$\mathcal{L} = \frac{1}{k_\mathrm{h}} = \frac{\lambda}{2\pi} \; . \tag{3.85}$$