Chapter 4

Equations of linear stellar oscillations

In the present chapter the equations governing small oscillations around a spherical equilibrium state are derived. The general equations were presented in Section 3.2. However, here we make explicit use of the spherical symmetry. These equations describe the general, so-called *nonradial* oscillations, where spherical symmetry of the perturbations is not assumed. The more familiar case of *radial*, or spherically symmetric, oscillations, is contained as a special case.

4.1 Mathematical preliminaries

It is convenient to write the solution to the perturbation equations on complex form, with the physically realistic solution being obtained as the real part of the complex solution. To see that this is possible, notice that the general equations can be written as

$$\mathbf{A} \frac{\partial \mathbf{y}}{\partial t} = \mathcal{B}(\mathbf{y}) \ , \tag{4.1}$$

where the vector $\mathbf{y}$ consists of the perturbation variables $(\boldsymbol{\delta}\mathbf{r}, p', \rho', \cdots)$, $\mathbf{A}$ is a matrix with real coefficients, and $\mathcal{B}$ is a linear matrix operator involving spatial gradients, *etc.*, with real coefficients. Neither $\mathbf{A}$ nor $\mathcal{B}$ depend on time. If $\mathbf{y}$ is a complex solution to equation (4.1) then the complex conjugate $\mathbf{y}^*$ is also a solution, since

$$\mathbf{A} \frac{\partial \mathbf{y}^*}{\partial t} = \left(\mathbf{A} \frac{\partial \mathbf{y}}{\partial t} \right)^* = [\mathcal{B}(\mathbf{y})]^* = \mathcal{B}(\mathbf{y}^*) \ , \tag{4.2}$$

and hence, as the system is linear and homogeneous, the real part $\Re(\mathbf{y}) = 1/2(\mathbf{y} + \mathbf{y}^*)$ is a solution.

Because of the independence of time of the coefficients in equation (4.1), solutions can be found of the form

$$\mathbf{y}(\mathbf{r}, t) = \hat{\mathbf{y}}(\mathbf{r}) \exp(-\, i \, \omega \, t) \ . \tag{4.3}$$

This is a solution if the *amplitude function* $\hat{\mathbf{y}}$ satisfies the eigenvalue equation

$$-i\omega \mathbf{A} \cdot \hat{\mathbf{y}} = \mathcal{B}(\hat{\mathbf{y}}) \ . \tag{4.4}$$

Equations of this form were also considered in Section 3.3 for simple waves. Note that in equations (4.3) and (4.4) the frequency ω must in general be assumed to be complex.

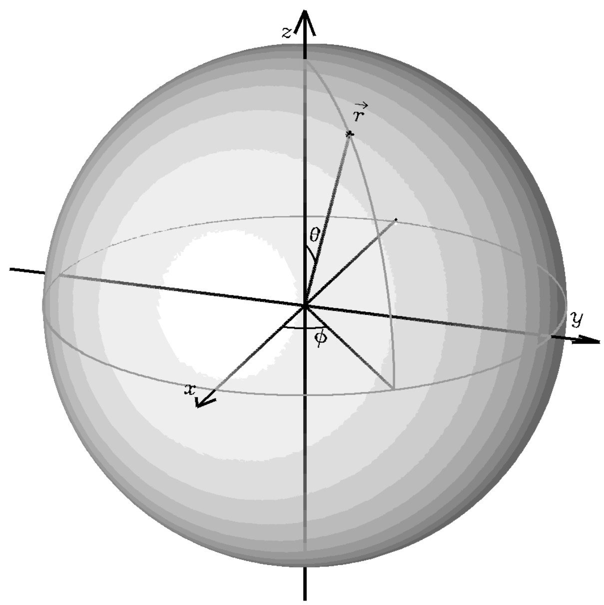

Figure 4.1: The spherical polar coordinate system.

Equation (4.3) is an example of the *separability* of the solution to a system of linear partial differential equations, when the equations do not depend of one of the coordinates. As the equilibrium state is spherically symmetric, we may expect a similar separability in spatial coordinates. Specifically I use spherical polar coordinates (r, θ, ϕ) (*cf.* Figure 4.1), where r is the distance to the centre, θ is colatitude (*i.e.*, the angle from the polar axis), and ϕ is longitude. Here the equilibrium is independent of θ and ϕ, and the solution must be separable. However, the form of the separated solution depends on the physical nature of the problem, and so must be discussed in the context of the reduction of the equations. This is done in the next section.

Here I present some relations in spherical polar coordinates that will be needed in the following [see also Appendix 2 of Batchelor (1967)]. Let $\mathbf{a}_r$, $\mathbf{a}_\theta$ and $\mathbf{a}_\phi$ be unit vectors in the r, θ and ϕ directions, let V be a general scalar field, and let

$$\mathbf{F} = F_r \mathbf{a}_r + F_\theta \mathbf{a}_\theta + F_\phi \mathbf{a}_\phi \tag{4.5}$$

be a vector field. Then the gradient of V is

$$\nabla V = \frac{\partial V}{\partial r}\mathbf{a}_r + \frac{1}{r}\frac{\partial V}{\partial \theta}\mathbf{a}_\theta + \frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi}\mathbf{a}_\phi \ , \tag{4.6}$$

the divergence of $\mathbf{F}$ is

$$\operatorname{div}\mathbf{F} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 F_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta F_\theta) + \frac{1}{r\sin\theta}\frac{\partial F_\phi}{\partial\phi} \ , \tag{4.7}$$

and consequently the Laplacian of V is

$$\begin{aligned}\nabla^2 V &= \operatorname{div}(\nabla V) \\ &= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 V}{\partial\phi^2} \ .\end{aligned} \tag{4.8}$$

Finally, we need the directional derivatives, in the direction, say, of the vector

$$\mathbf{n} = n_r\mathbf{a}_r + n_\theta\mathbf{a}_\theta + n_\phi\mathbf{a}_\phi. \tag{4.9}$$

The directional derivative $\mathbf{n}\cdot\nabla V$ of a scalar is obtained, as would be naively expected, as the scalar product of $\mathbf{n}$ with the gradient in equation (4.6). However, in the directional derivatives $\mathbf{n}\cdot\nabla\mathbf{F}$ of a vector field, the change in the unit vectors $\mathbf{a}_r$, $\mathbf{a}_\theta$ and $\mathbf{a}_\phi$ must be taken into account. The result is

$$\begin{aligned}\mathbf{n}\cdot\nabla\mathbf{F} &= \left(\mathbf{n}\cdot\nabla F_r - \frac{n_\theta F_\theta}{r} - \frac{n_\phi F_\phi}{r}\right)\mathbf{a}_r \\ &\quad + \left(\mathbf{n}\cdot\nabla F_\theta - \frac{n_\phi F_\phi}{r}\cot\theta + \frac{n_\theta F_r}{r}\right)\mathbf{a}_\theta \\ &\quad + \left(\mathbf{n}\cdot\nabla F_\phi + \frac{n_\phi F_r}{r} + \frac{n_\phi F_\theta}{r}\cot\theta\right)\mathbf{a}_\phi \ ,\end{aligned} \tag{4.10}$$

where the directional derivatives of F_r, F_θ and F_ϕ are the same as for a scalar field.

As the radial direction has a special status, it is convenient to introduce the horizontal (or, properly speaking, tangential) component of the vector $\mathbf{F}$:

$$\mathbf{F}_{\mathrm{h}} = F_\theta\mathbf{a}_\theta + F_\phi\mathbf{a}_\phi \ , \tag{4.11}$$

and similarly the horizontal components of the gradient, divergence and Laplacian as

$$\nabla_{\mathrm{h}} V = \frac{1}{r}\frac{\partial V}{\partial\theta}\mathbf{a}_\theta + \frac{1}{r\sin\theta}\frac{\partial V}{\partial\phi}\mathbf{a}_\phi \ , \tag{4.12}$$

$$\nabla_{\mathrm{h}}\cdot\mathbf{F} = \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta F_\theta) + \frac{1}{r\sin\theta}\frac{\partial F_\phi}{\partial\phi} \ , \tag{4.13}$$

and

$$\nabla^2_{\mathrm{h}} V = \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial V}{\partial\theta}) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 V}{\partial\phi^2} \ . \tag{4.14}$$

4.2 The Oscillation Equations

4.2.1 Separation of variables

The displacement $\boldsymbol{\delta}\mathbf{r}$ is separated into radial and horizontal components as

$$\boldsymbol{\delta}\mathbf{r} = \xi_r \mathbf{a}_r + \boldsymbol{\xi}_\mathrm{h} \; . \tag{4.15}$$

The horizontal component of the equations of motion, (3.43), is

$$\rho_0 \frac{\partial^2 \boldsymbol{\xi}_\mathrm{h}}{\partial t^2} = -\nabla_\mathrm{h} p' - \rho_0 \nabla_\mathrm{h} \Phi' \; . \tag{4.16}$$

As the horizontal gradient of equilibrium quantities is zero, the horizontal divergence of equation (4.16) gives

$$\rho_0 \frac{\partial^2}{\partial t^2} \nabla_\mathrm{h} \cdot \boldsymbol{\xi}_\mathrm{h} = -\nabla_\mathrm{h}^2 p' - \rho_0 \nabla_\mathrm{h}^2 \Phi' \; . \tag{4.17}$$

The equation of continuity, (3.41), can be written as

$$\rho' = -\frac{1}{r^2} \frac{\partial}{\partial r} (\rho_0 r^2 \xi_r) - \rho_0 \nabla_\mathrm{h} \cdot \boldsymbol{\xi}_\mathrm{h} \; . \tag{4.18}$$

This can be used to eliminate $\nabla_\mathrm{h} \cdot \boldsymbol{\xi}_\mathrm{h}$ from equation (4.17), which becomes

$$-\frac{\partial^2}{\partial t^2} \left[\rho' + \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 \rho_0 \xi_r) \right] = -\nabla_\mathrm{h}^2 p' - \rho_0 \nabla_\mathrm{h}^2 \Phi' \; . \tag{4.19}$$

The radial component of equation (3.43) is

$$\rho_0 \frac{\partial^2 \xi_r}{\partial t^2} = -\frac{\partial p'}{\partial r} - \rho' g_0 - \rho_0 \frac{\partial \Phi'}{\partial r} \; . \tag{4.20}$$

Finally, Poisson's equation (3.44) may be written as

$$\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial \Phi'}{\partial r} \right) + \nabla_\mathrm{h}^2 \Phi' = 4\pi G \rho' \; . \tag{4.21}$$

It should be noticed that in equations (4.19) – (4.21) derivatives with respect to the angular variables θ and ϕ only appear in the combination ∇_h^2.

We now have to consider the energy equation (3.47), together with equation (3.48) for the heat gain. The result clearly depends on the form assumed for the flux $\mathbf{F}$. However, if the flux can be expressed in terms of a gradient of a scalar, as in the diffusion approximation [equation (3.22)], the energy equation also only contains derivatives with respect to θ and ϕ in ∇_h^2.

Exercise 4.1:

Show this.

We may now address the separation of the angular variables. The object is to factor out the variation of the perturbations with θ and ϕ as a function $f(\theta, \phi)$. From the form of the equations this is clearly possible, if f is an eigenfunction of the horizontal Laplace operator,

$$\nabla_{\rm h}^2 f = -\frac{1}{r^2}\Lambda f \; , \tag{4.22}$$

where Λ is a constant. That $1/r^2$ has to appear is obvious from equation (4.14); the choice of sign is motivated later. Writing it out in full, equation (4.22) becomes

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 f}{\partial\phi^2} = -\Lambda\, f \; . \tag{4.23}$$

As the coefficients in this equation are independent of ϕ, the solution can be further separated, as

$$f(\theta,\phi) = f_1(\theta) f_2(\phi) \; . \tag{4.24}$$

It follows from equation (4.23) that f_2 satisfies an equation of the form

$$\frac{{\rm d}^2 f_2}{{\rm d}\phi^2} = \alpha f_2 \; , \tag{4.25}$$

where α is another constant; this has the solution $f_2 = \exp(\pm\alpha^{1/2}\phi)$. However, the solution has to be continuous and hence periodic, *i.e.*, $f_2(0) = f_2(2\pi)$. Consequently we must demand that $\alpha^{1/2} = im$, where m is an integer.

When used in equation (4.23), this gives the following differential equation for f_1:

$$\frac{\rm d}{{\rm d}x}\left[(1-x^2)\frac{{\rm d}f_1}{{\rm d}x}\right] + \left(\Lambda - \frac{m^2}{1-x^2}\right) f_1 = 0 \; , \tag{4.26}$$

where $x = \cos\theta$. It can be shown that this equation has a regular solution only when

$$\Lambda = l(l+1) \; , \tag{4.27}$$

where l is a non-negative integer and

$$|m| \le l \; . \tag{4.28}$$

The regular solution is

$$f_1(\theta) = P_l^m(\cos\theta) \; , \tag{4.29}$$

where P_l^m is the Legendre function. By including an appropriate scaling factor we may finally write

$$f(\theta,\phi) = (-1)^m c_{lm} P_l^m(\cos\theta)\exp(im\phi) \equiv Y_l^m(\theta,\phi) \; , \tag{4.30}$$

where Y_l^m is a spherical harmonic; here c_{lm} is a normalization constant given by equation (2.2), such that the integral of $|Y_l^m|^2$ over the unit sphere is 1. Y_l^m is characterized by its *degree* l and its azimuthal order m; the properties of spherical harmonics were discussed in more detail in Section 2.1 (see also Appendix A). From equations (4.22) and (4.27) we also have that

$$\nabla_{\rm h}^2 f = -\frac{l(l+1)}{r^2} f \; . \tag{4.31}$$

The dependent variables in equations (4.19) – (4.21) can now be written as

$$\xi_r(r,\theta,\phi,t) = \sqrt{4\pi}\,\tilde\xi_r(r) Y_l^m(\theta,\phi)\exp(-i\omega t) \; , \tag{4.32}$$

$$p'(r,\theta,\phi,t) = \sqrt{4\pi}\,\tilde{p}'(r) Y_l^m(\theta,\phi)\exp(-i\omega t) \; , \tag{4.33}$$

etc. Also it follows from equation (3.38) that if the Eulerian perturbations are on the form given in these equations, so are the Lagrangian perturbations. Then the equations contain $Y_l^m(\theta,\phi)\exp(-i\omega t)$ as a common factor. After dividing by it, the following ordinary differential equations for the amplitude functions $\tilde{\xi}_r, \tilde{p}', \cdots$, result:

$$\omega^2\left[\tilde{\rho}' + \frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}(r^2\rho_0\tilde{\xi}_r)\right] = \frac{l(l+1)}{r^2}(\tilde{p}' + \rho_0\tilde{\Phi}') \; , \tag{4.34}$$

$$-\omega^2\rho_0\tilde{\xi}_r = -\frac{\mathrm{d}\tilde{p}'}{\mathrm{d}r} - \tilde{\rho}' g_0 - \rho_0\frac{\mathrm{d}\tilde{\Phi}'}{\mathrm{d}r} \; , \tag{4.35}$$

$$\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}\tilde{\Phi}'}{\mathrm{d}r}\right) - \frac{l(l+1)}{r^2}\tilde{\Phi}' = 4\pi G\tilde{\rho}' \; , \tag{4.36}$$

together with the energy equation

$$\left(\delta\tilde{p} - \frac{\Gamma_{1,0}p_0}{\rho_0}\delta\tilde{\rho}\right) = \rho_0(\Gamma_{3,0}-1)\delta\tilde{q} \; . \tag{4.37}$$

It should be noted that equations (4.34) – (4.37) do not depend on the azimuthal order m. This is a consequence of the assumed spherical symmetry of the equilibrium state, which demands that the results should be independent of the choice of polar axis for the coordinate system. Changing the polar axis would change the spherical harmonics, in such a way that a new spherical harmonic, with given l and m, would be a linear combination over m of the old spherical harmonics with the given value of l (Edmonds 1960). As this change of axis can have no effect on the dynamics of the oscillations, the equations must be independent of m, as found here.

From equation (4.16) the horizontal component of the displacement is given by

$$\boldsymbol{\xi}_\mathrm{h} = \sqrt{4\pi}\,\tilde{\xi}_\mathrm{h}(r)\left(\frac{\partial Y_l^m}{\partial\theta}\mathbf{a}_\theta + \frac{1}{\sin\theta}\frac{\partial Y_l^m}{\partial\phi}\mathbf{a}_\phi\right)\exp(-i\omega t) \; , \tag{4.38}$$

where

$$\tilde{\xi}_\mathrm{h}(r) = \frac{1}{r\omega^2}(\frac{1}{\rho_0}\tilde{p}' + \tilde{\Phi}') \; . \tag{4.39}$$

Thus the (physical) displacement vector can be written as

$$\begin{aligned}\boldsymbol{\delta}\mathbf{r} \;=\; & \sqrt{4\pi}\Re\Big\{\Big[\tilde{\xi}_r(r)Y_l^m(\theta,\phi)\mathbf{a}_r \\ & +\tilde{\xi}_\mathrm{h}(r)\left(\frac{\partial Y_l^m}{\partial\theta}\mathbf{a}_\theta + \frac{1}{\sin\theta}\frac{\partial Y_l^m}{\partial\phi}\mathbf{a}_\phi\right)\Big]\exp(-i\omega t)\Big\} \; .\end{aligned} \tag{4.40}$$

As noted in Section 4.1 the frequency ω is in general complex. That this is so may be seen from the energy equation (4.37), if the expression (3.48) for the heating rate perturbation is used. Assuming the time dependence given in equations (4.33) for the perturbed quantities, equation (3.48) can be written as

$$\delta q = \frac{i}{\rho_0\omega}\delta(\rho\epsilon - \mathrm{div}\,\mathbf{F}) \; . \tag{4.41}$$

Here the perturbations on the right-hand side can be expressed in terms of the perturbations in, say, density and temperature. For instance, since ϵ is a function $\epsilon(\rho, T)$ of density and temperature, we obtain

$$\delta(\rho\epsilon) = \rho\epsilon \left\{ \left[1 + \left(\frac{\partial \ln \epsilon}{\partial \ln \rho} \right)_T \right] \frac{\delta\rho}{\rho} + \left(\frac{\partial \ln \epsilon}{\partial \ln T} \right)_\rho \frac{\delta T}{T} \right\} . \tag{4.42}$$

The expression for $\delta(\operatorname{div} \mathbf{F})$ depends on the treatment of the energy transport, discussed in Section 3.1.3. Often the diffusion approximation is adequate; then $\delta(\operatorname{div} \mathbf{F})$ may be obtained in a fashion similar to the derivation of equation (4.42) by perturbing equation (3.22), although with considerable effort. Note that this gives rise to a term in the second derivative of δT with respect to r; the same is true if the Eddington approximation [equation (3.24)] is used, whereas the use of Newton's law of cooling [equation (3.23)] gives a direct relation between the heat loss and the local thermodynamic variables, and hence does not increase the order of the equations. However, regardless of the approximation used, substitution of the relevant relations into the energy equation, written in terms of ρ and T, results in an equation which, because of the factor i/ω in the expression for δq, has complex coefficients. Hence the oscillation equations cannot in general have a real solution.

The complex frequency can be expressed as $\omega = \omega_\mathrm{r} + i\eta$, where ω_r and η are real; consequently the dependence of the perturbations on ϕ and t is of the form

$$\cos(m\phi - \omega_\mathrm{r} t + \delta_0) \mathrm{e}^{\eta t} , \tag{4.43}$$

where δ_0 is the initial phase. For $m \neq 0$ this describes a wave traveling around the equator with angular phase speed ω_r/m, whereas for $m = 0$ the perturbation is a standing wave. The period of the perturbation is $\Pi = 2\pi/\omega_\mathrm{r}$. Its amplitude grows or decays exponentially with time, depending on whether the *growth rate* η is positive or negative.

Neglecting η, we may obtain the mean square components of the displacement, when averaged over a spherical surface and time, from equation (4.40). For the radial component the result is

$$\begin{aligned} \delta r_\mathrm{rms}^2 &= \langle |\delta \mathbf{r} \cdot \mathbf{a}_r|^2 \rangle \qquad (4.44) \\ &= \frac{1}{\Pi} \int_0^\Pi \mathrm{d}t \frac{1}{4\pi} \oint \left\{ \Re \left[\tilde{\xi}_r(r) Y_l^m(\theta, \phi) \exp(-i\omega t) \right] \right\}^2 \mathrm{d}\Omega \\ &= \frac{1}{2} |\tilde{\xi}_r(r)|^2 , \end{aligned}$$

where Ω is solid angle. Similarly, the mean square length of the horizontal component of $\boldsymbol{\delta}\mathbf{r}$ is

$$\delta h_\mathrm{rms}^2 = \langle |\boldsymbol{\xi}_\mathrm{h}|^2 \rangle = 1/2\, l(l+1) |\tilde{\xi}_\mathrm{h}(r)|^2 , \tag{4.45}$$

where $\tilde{\xi}_\mathrm{h}$ is the amplitude function introduced in equation (4.39).

Exercise 4.2:

Verify equations (4.44) and (4.45). Note that the latter is a little tricky: this requires integration by parts and use of the fact that P_l^m satisfies equation (4.23).

The kinetic energy of pulsation is

$$E_{\rm kin} = \frac{1}{2} \int_V |\mathbf{v}|^2 \rho_0 \mathrm{d}V \; . \tag{4.46}$$

As in equations (4.44) and (4.45) it follows from equation (4.40) that the time-averaged energy is $1/4\,\omega^2 \mathcal{E}$, where

$$\mathcal{E} = 4\pi \int_0^R [|\tilde{\xi}_r(r)|^2 + l(l+1)|\tilde{\xi}_{\rm h}(r)|^2]\, \rho_0 r^2 \mathrm{d}r \; . \tag{4.47}$$

For $m \neq 0$ $E_{\rm kin}$ is independent of t, in accordance with the running-wave nature of the oscillation in this case, whereas for $m = 0$ we have $E_{\rm kin} = \frac{1}{2}\omega^2 \mathcal{E} \cos^2(\omega t - \delta_0)$. It is convenient to introduce the dimensionless measure E of $\mathcal{E}$, by

$$E = \frac{4\pi \int_0^R [|\tilde{\xi}_r(r)|^2 + l(l+1)|\tilde{\xi}_{\rm h}(r)|^2] \rho_0 r^2 \mathrm{d}r}{M\,[|\tilde{\xi}_r(R)|^2 + l(l+1)|\tilde{\xi}_{\rm h}(R)|^2]} = \frac{M_{\rm mode}}{M} \; , \tag{4.48}$$

where M is the total mass of the star, and $M_{\rm mode}$ is the so-called modal mass; thus E provides a measure of the normalized *inertia* of the mode. These quantities are defined such that the time-averaged kinetic energy in the oscillation is

$$1/2 M_{\rm mode} V_{\rm rms}^2 = 1/2 E M V_{\rm rms}^2 \; , \tag{4.49}$$

where $V_{\rm rms}^2$ is the mean, over the stellar surface and time, of the squared total velocity of the mode.

From equation (4.31) it follows that for any perturbation quantity ψ',

$$\nabla_{\rm h}^2 \psi' = -\frac{l(l+1)}{r^2} \psi' \; . \tag{4.50}$$

Thus if the oscillations are regarded locally as plane waves, as in equation (3.53), we may make the identification

$$\frac{l(l+1)}{r^2} = k_{\rm h}^2 \; , \tag{4.51}$$

where $k_{\rm h}$ is the length of the horizontal component of the wave vector, as in equation (3.63); note in particular that $k_{\rm h}$ depends on r.

For completeness, I note that the modes discussed so far (which are the only modes considered in the following), are known as *spheroidal modes*. In addition there is a second class of modes, the *toroidal modes*, which are briefly discussed in Cox (1980), Section 17.3. In a spherically symmetric (and hence nonrotating) star, they have zero frequency and correspond to infinitely slow, purely horizontal motion. In a rotating star they give rise to oscillations whose frequencies are of the order of the rotation frequency.

4.2.2 Radial oscillations

For *radial* oscillations, with $l = 0$, the perturbation in the gravitational field may be eliminated analytically. From Poisson's equation in the form (4.36) we have, by using the equation of continuity (4.18) with zero horizontal part, that

$$\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2 \frac{\mathrm{d}\tilde{\Phi}'}{\mathrm{d}r}\right) = -\frac{4\pi G}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}(r^2 \rho_0 \tilde{\xi}_r) \; , \tag{4.52}$$

or, as the gravitational force must be finite at $r = 0$,

$$\frac{\mathrm{d}\tilde{\Phi}'}{\mathrm{d}r} = -4\pi G \rho_0 \tilde{\xi}_r \; . \tag{4.53}$$

Furthermore, the term containing $\tilde{\Phi}'$ drops out in equation (4.34).

With these eliminations, the oscillation equations can be reduced to a relatively simple form. We write the energy equation (4.37) as

$$\tilde{\rho}' = \frac{\rho_0}{\Gamma_{1,0} p_0} \tilde{p}' + \rho_0 \tilde{\xi}_r \left(\frac{1}{\Gamma_{1,0} p_0} \frac{\mathrm{d}p_0}{\mathrm{d}r} - \frac{1}{\rho_0} \frac{\mathrm{d}\rho_0}{\mathrm{d}r} \right) - \frac{\rho_0^2}{\Gamma_{1,0} p_0} (\Gamma_{3,0} - 1) \delta \tilde{q} \; . \tag{4.54}$$

Then equation (4.34) may be written as

$$\frac{\mathrm{d}\tilde{\xi}_r}{\mathrm{d}r} = -\frac{2}{r} \tilde{\xi}_r - \frac{1}{\Gamma_{1,0} p_0} \frac{\mathrm{d}p_0}{\mathrm{d}r} \tilde{\xi}_r - \frac{1}{\Gamma_{1,0} p_0} \tilde{p}' + \frac{\rho_0}{\Gamma_{1,0} p_0} (\Gamma_{3,0} - 1) \delta \tilde{q} \; , \tag{4.55}$$

or, introducing $\zeta \equiv \tilde{\xi}_r / r$,

$$\tilde{p}' = -\Gamma_{1,0} p_0 r \left(\frac{\mathrm{d}\zeta}{\mathrm{d}r} + \frac{3}{r}\zeta + \frac{1}{\Gamma_{1,0} p_0} \frac{\mathrm{d}p_0}{\mathrm{d}r} \zeta \right) + \rho_0 (\Gamma_{3,0} - 1) \delta \tilde{q} \; . \tag{4.56}$$

By substituting equations (4.53), (4.54) and (4.56) into equation (4.35) we obtain, after a little manipulation,

$$\frac{1}{r^3} \frac{\mathrm{d}}{\mathrm{d}r} \left(r^4 \Gamma_{1,0} p_0 \frac{\mathrm{d}\zeta}{\mathrm{d}r} \right) + \frac{\mathrm{d}}{\mathrm{d}r} [(3\Gamma_{1,0} - 4) p_0] \zeta + \rho_0 \omega^2 r \zeta = \frac{\mathrm{d}}{\mathrm{d}r} [\rho_0 (\Gamma_{3,0} - 1) \delta \tilde{q}] \; . \tag{4.57}$$

Exercise 4.3:

Fill in the missing steps in the derivation of equation (4.57).

It is important to note that the apparent simplicity of equation (4.57) hides a great deal of complexity in the heating term on the right-hand side. Nevertheless, this equation is convenient for discussions of the properties of radial oscillations. In these notes, however, I shall mostly consider the general equations for *nonradial* oscillations, where l can take any value.

4.3 Linear, adiabatic oscillations

To simplify the notation, from now on I drop the tilde on the amplitude functions, and the "0" on equilibrium quantities. This should not cause any confusion.

4.3.1 Equations

For adiabatic oscillations, $\delta q = 0$ and equation (4.37) can be written

$$\rho' = \frac{\rho}{\Gamma_1 p} p' + \rho \xi_r \left(\frac{1}{\Gamma_1 p} \frac{\mathrm{d}p}{\mathrm{d}r} - \frac{1}{\rho} \frac{\mathrm{d}\rho}{\mathrm{d}r} \right) . \tag{4.58}$$

This may be used to eliminate ρ' from equations (4.34) – (4.36). From equation (4.34) we obtain

$$\frac{\mathrm{d}\xi_r}{\mathrm{d}r} = -\left(\frac{2}{r} + \frac{1}{\Gamma_1 p} \frac{\mathrm{d}p}{\mathrm{d}r} \right) \xi_r + \frac{1}{\rho} \left[\frac{l(l+1)}{\omega^2 r^2} - \frac{1}{c^2} \right] p' + \frac{l(l+1)}{\omega^2 r^2} \Phi' , \tag{4.59}$$

where we used that $c^2 = \Gamma_1 p/\rho$ is the square of the adiabatic sound speed [*cf.* equation (3.52)]. It is convenient to introduce the characteristic acoustic frequency S_l by

$$S_l^2 = \frac{l(l+1)c^2}{r^2} = k_\mathrm{h}^2 c^2 . \tag{4.60}$$

Then equation (4.59) can be written as

$$\frac{\mathrm{d}\xi_r}{\mathrm{d}r} = -\left(\frac{2}{r} + \frac{1}{\Gamma_1 p} \frac{\mathrm{d}p}{\mathrm{d}r} \right) \xi_r + \frac{1}{\rho c^2} \left(\frac{S_l^2}{\omega^2} - 1 \right) p' + \frac{l(l+1)}{\omega^2 r^2} \Phi' . \tag{4.61}$$

Equation (4.35) gives

$$\frac{\mathrm{d}p'}{\mathrm{d}r} = \rho(\omega^2 - N^2)\xi_r + \frac{1}{\Gamma_1 p} \frac{\mathrm{d}p}{\mathrm{d}r} p' - \rho \frac{\mathrm{d}\Phi'}{\mathrm{d}r} , \tag{4.62}$$

where, as in equation (3.73), N is the buoyancy frequency, given by

$$N^2 = g \left(\frac{1}{\Gamma_1 p} \frac{\mathrm{d}p}{\mathrm{d}r} - \frac{1}{\rho} \frac{\mathrm{d}\rho}{\mathrm{d}r} \right) . \tag{4.63}$$

Finally, equation (4.36) becomes

$$\frac{1}{r^2} \frac{\mathrm{d}}{\mathrm{d}r} \left(r^2 \frac{\mathrm{d}\Phi'}{\mathrm{d}r} \right) = 4\pi G \left(\frac{p'}{c^2} + \frac{\rho \xi_r}{g} N^2 \right) + \frac{l(l+1)}{r^2} \Phi' . \tag{4.64}$$

Equations (4.61), (4.62) and (4.64) constitute a fourth-order system of ordinary differential equations for the four dependent variables ξ_r, p', Φ' and $d\Phi'/\mathrm{d}r$. Thus it is a complete set of differential equations.

For radial oscillations equations (4.61) and (4.62), after elimination of the terms in Φ' by means of equation (4.53), reduce to a second-order system in ξ_r and p'; an alternative formulation of this set of equations is obtained from equation (4.57), by setting the right-hand side to zero. The reduction to second order is a useful simplification from a computational point of view, and it may be exploited in asymptotic analyses. However, here I shall always treat radial oscillations in the same way as the nonradial case.

It should be noticed that all coefficients in equations (4.61), (4.62) and (4.64) are real. Also, as discussed below, the same is true of the boundary conditions. Since the frequency only appears in the form ω^2, we may expect that the solution is such that ω^2 is real, in which case the eigenfunctions may also be chosen to be real. This may be proved to be true in general. Thus the frequency is either purely real, in which case the motion is an undamped oscillator, or purely imaginary, so that the motion grows or decays exponentially. From a physical point of view this results from the adiabatic approximation, which ensures that energy cannot be fed into the motion, except from the gravitational field; thus the only possible type of instability is a dynamical instability. I shall almost always consider the oscillatory case, with $\omega^2 > 0$; note, however, that the convective instability discussed briefly in Section 3.3.2 is an example of a dynamical instability.

4.3.2 Boundary conditions

To supplement the four equations in the general case, we need four boundary conditions. These are discussed in considerable detail in Unno *et al.* (1989), Section 18.1, and in Cox (1980), Section 17.6. Here I give only a brief summary.

The centre, $r = 0$ is a regular singular point of the equations. Thus, as is usual in the theory of differential equations, the equations admit both regular and singular solutions at this point. Two of the conditions serve to select the regular solutions. By expanding the equations, it may be shown that near $r = 0$, ξ_r behaves like r^{l-1}, whereas p' and Φ' behave as r^l. In the special case of radial oscillations, however, the coefficient to the leading-order term in ξ_r vanishes, and ξ_r goes as r. Indeed it is obvious from geometrical considerations that for spherically symmetric oscillations, the displacement must vanish at the centre. From the expansions, two relations between the solution near $r = 0$ may be obtained. In particular, it may be shown that for $l > 0$,

$$\xi_r \simeq l \xi_{\rm h} \; , \qquad \text{for } r \to 0 \; . \tag{4.65}$$

In the radial case, one of the conditions was implicitly used to obtain equation (4.53), and only one central condition remains.

One surface condition is obtained by demanding continuity of Φ' and its derivative at the surface radius $r = R$. Outside the star the density perturbation vanishes, and Poisson's equation may be solved analytically. The solution vanishing at infinity is

$$\Phi' = A \, r^{-l-1} \; , \tag{4.66}$$

where A is a constant. Consequently Φ' must satisfy

$$\frac{{\rm d}\Phi'}{{\rm d}r} + \frac{l+1}{r} \Phi' = 0 \qquad \text{at } r = R \; . \tag{4.67}$$

The second condition depends on the treatment of the stellar atmosphere, and may consequently be quite complicated. These complications are discussed further in Chapter 5. For the moment, I note that if the star is assigned a definite boundary at $r = R$, it is physically reasonable to assume that the boundary is free, so that no forces act on it. In this way the star can be considered as an isolated system. This is equivalent to requiring the pressure to be constant at the perturbed surface. Thus as the second surface boundary condition I take that the Lagrangian pressure perturbation vanish, *i.e.*,

$$\delta p = p' + \xi_r \frac{{\rm d}p}{{\rm d}r} = 0 \qquad \text{at } r = R \; . \tag{4.68}$$

As shown later, a more detailed analysis of the atmospheric behaviour of the oscillations gives a very similar result, except at high frequencies.

From equation (4.68) one can estimate the ratio between the radial and horizontal components of the displacement on the surface. The amplitude of the horizontal displacement is given by equation (4.39). In most cases, however, the perturbation in the gravitational potential is small. Thus we have approximately, from equation (4.68), that

$$\frac{\xi_{\rm h}(R)}{\xi_r(R)} = \frac{g_{\rm s}}{R\omega^2} \equiv \sigma^{-2} \; , \tag{4.69}$$

where $g_{\rm s}$ is the surface gravity, and σ is a dimensionless frequency, defined by

$$\sigma^2 = \frac{R^3}{GM}\omega^2 \; . \tag{4.70}$$

Thus the surface value of $\xi_{\rm h}/\xi_r$, to this approximation, depends only on frequency. The ratio of the *rms* horizontal to vertical displacement [*cf.* equations (4.44) and (4.45)] is

$$\frac{\delta h_{\rm rms}}{\delta r_{\rm rms}} = \frac{\sqrt{l(l+1)}}{\sigma^2} \qquad \text{at } r = R \; . \tag{4.71}$$

For the important case of the solar five-minute oscillations, $\sigma^2 \sim 1000$, and so the motion is predominantly vertical, except at large l.