Chapter 8

Rotation and stellar oscillations

I have so far assumed that there are no velocity fields in the equilibrium model. This is manifestly false for an object like the Sun which is rotating; in particular, the observed surface rotation depends on latitude, thus implying the presence of velocity fields. In addition, other large-scale velocity fields, such as those caused by convection, could have an effect on the modes. Hence we must investigate such effects. Apart from their intrinsic interest, the principal purpose of such studies is obviously to be able to probe the velocity fields from the observed properties of the oscillations.

It is straightforward to see, from a purely geometrical argument, that rotation might affect the observed frequencies. Assume the angular velocity Ω to be uniform, and consider an oscillation with a frequency ω_0, independent of m, in the frame rotating with the star. I introduce a coordinate system in this frame, with coordinates (r', θ', ϕ') which are related to the coordinates (r, θ, ϕ) in an inertial frame through

$$(r', \theta', \phi') = (r, \theta, \phi - \Omega t) \ . \tag{8.1}$$

It follows from equation (4.40) that, in the rotating frame, the perturbations depend on ϕ' and t as $\cos(m\phi' - \omega_0 t)$; hence, the dependence in the inertial frame is $\cos(m\phi - \omega_m t)$, where

$$\omega_m = \omega_0 + m\Omega \ . \tag{8.2}$$

Thus an observer in the inertial frame finds that the frequency is split uniformly according to m.

This description is obviously incomplete. Even in the case of uniform rotation, the effects of the Coriolis force must be taken into account in the rotating frame, causing a contribution to the frequency splitting (Cowling and Newing 1949; Ledoux 1949). Furthermore, in general the angular velocity is a function $\Omega(r, \theta)$ of position. Nevertheless, as shown below, the effect of the Coriolis force is often small and equation (8.2) is approximately correct if Ω is replaced by a suitable average of the position-dependent angular velocity.

To arrive at an expression valid for any rotation law it is convenient to consider first the even more general case of an arbitrary stationary velocity field in the star.

8.1 The effect of large-scale velocities on the oscillation frequencies

We need to reconsider the derivation of the perturbation equations, including the effects of a velocity field. I shall assume that the equilibrium structure is stationary, so that all local time derivatives vanish. Even with this assumption the determination of the equilibrium structure is a non-trivial problem, due to the distortion caused by the velocity fields (*e.g.* due to centrifugal effects in a rotating star). However, here I assume that the velocity $\mathbf{v}_0$ in the equilibrium state is sufficiently slow that terms quadratic in $\mathbf{v}_0$ can be neglected. The equation of continuity (3.6) gives, because of the assumed stationarity

$$\mathrm{div}\,(\rho_0 \mathbf{v}_0) = 0 \; . \qquad (8.3)$$

Also, because of the neglect of terms of order $|\mathbf{v}|^2$, the equations of motion (3.9) reduce to

$$0 = -\nabla p_0 + \rho_0 \mathbf{g}_0 \; . \qquad (8.4)$$

As usual I have replaced the body force per unit mass $\mathbf{f}$ by the gravitational acceleration $\mathbf{g}$. Thus equation (3.30) of hydrostatic support is unchanged. In the solar case, the ratio between the neglected centrifugal force and surface gravity is of order 2×10^{-5} and so the error in equation (8.4) is in fact small.

The perturbation analysis also requires some care. It was treated in considerable detail by Lynden-Bell & Ostriker (1967), and is discussed in Cox (1980), Chapter 5. Here I just present a few of the main features.

The velocity at a given point in space can be written as

$$\mathbf{v} = \mathbf{v}_0 + \mathbf{v}' \; , \qquad (8.5)$$

where $\mathbf{v}'$ is the Eulerian velocity perturbation. The displacement $\boldsymbol{\delta}\mathbf{r}$ must be determined relative to the moving equilibrium fluid; it is related to the velocity perturbation by

$$\frac{\mathrm{d}\boldsymbol{\delta}\mathbf{r}}{\mathrm{d}t} = \boldsymbol{\delta}\mathbf{v} = \mathbf{v}' + (\boldsymbol{\delta}\mathbf{r} \cdot \nabla)\mathbf{v}_0 \; . \qquad (8.6)$$

Here $\boldsymbol{\delta}\mathbf{v}$ is the Lagrangian velocity perturbation and, as in Section 3.1, $\mathrm{d}/\mathrm{d}t$ is the material time derivative,

$$\frac{\mathrm{d}\boldsymbol{\delta}\mathbf{r}}{\mathrm{d}t} = \frac{\partial \boldsymbol{\delta}\mathbf{r}}{\partial t} + (\mathbf{v}_0 \cdot \nabla)\boldsymbol{\delta}\mathbf{r} \; ; \qquad (8.7)$$

in contrast to the zero-velocity case, the local and the material time derivatives of perturbations are now different.

The perturbed continuity equation may be written as

$$\begin{aligned} 0 &= \frac{\partial \rho'}{\partial t} + \mathrm{div}\,(\rho' \mathbf{v}_0 + \rho_0 \mathbf{v}') \qquad (8.8) \\ &= \frac{\partial}{\partial t}[\rho' + \mathrm{div}\,(\rho_0 \boldsymbol{\delta}\mathbf{r})] + \mathrm{div}\,\{\rho' \mathbf{v}_0 + \rho_0[(\mathbf{v}_0 \cdot \nabla)\boldsymbol{\delta}\mathbf{r} - (\boldsymbol{\delta}\mathbf{r} \cdot \nabla)\mathbf{v}_0]\} \; , \end{aligned}$$

on using equations (8.6) and (8.7). After some manipulation, using equation (8.3), this may be reduced to

$$\frac{\partial A}{\partial t} + \mathrm{div}\,(A \mathbf{v}_0) = 0 \; , \qquad (8.9)$$

where

$$A = \rho' + \mathrm{div}\,(\rho_0 \boldsymbol{\delta}\mathbf{r}) \; . \tag{8.10}$$

This may also, by using again equation (8.3), be written as

$$\rho_0 \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{A}{\rho_0}\right) = 0 \; , \tag{8.11}$$

from which we conclude that $A = 0$, *i.e.*, that equation (3.41) is valid in this case also.

To obtain the perturbed momentum equation I use equation (3.8); from the fact that Lagrangian perturbation and material time derivative commute,

$$\frac{\mathrm{d}}{\mathrm{d}t}(\delta\psi) = \delta\left(\frac{\mathrm{d}\psi}{\mathrm{d}t}\right) \tag{8.12}$$

for any quantity ψ, we then obtain

$$\rho_0 \frac{\mathrm{d}\boldsymbol{\delta}\mathbf{v}}{\mathrm{d}t} = \delta(-\nabla p + \rho \mathbf{g}) = -\nabla p' + \rho_0 \mathbf{g}' + \rho' \mathbf{g}_0 \; , \tag{8.13}$$

by using equation (8.4). Alternatively this may be written, from equation (8.6), as

$$\rho_0 \frac{\mathrm{d}^2 \boldsymbol{\delta}\mathbf{r}}{\mathrm{d}t^2} = -\nabla p' + \rho_0 \mathbf{g}' + \rho' \mathbf{g}_0 \; , \tag{8.14}$$

or, by using equation (8.7) and neglecting the term quadratic in $\mathbf{v}_0$,

$$\rho_0 \frac{\partial^2 \boldsymbol{\delta}\mathbf{r}}{\partial t^2} + 2\rho_0 (\mathbf{v}_0 \cdot \nabla)\left(\frac{\partial \boldsymbol{\delta}\mathbf{r}}{\partial t}\right) = -\nabla p' + \rho_0 \mathbf{g}' + \rho' \mathbf{g}_0 \; , \tag{8.15}$$

which replaces equation (3.43). Finally, from the commutativity in equation (8.12), one finds that the perturbed energy equation (3.46) is still valid. Thus to this level of accuracy, the only change in the perturbation equations is the inclusion of the term in the first time derivative of $\boldsymbol{\delta}\mathbf{r}$ in equation (8.15)

As the equilibrium structure is independent of time, we may still separate the time dependence as $\exp(-i\omega t)$. Using, for simplicity, the same symbols for the amplitude functions in this separation, we obtain from the equations of motion

$$-\omega^2 \rho_0 \boldsymbol{\delta}\mathbf{r} - 2i\omega\rho_0 (\mathbf{v}_0 \cdot \nabla)\boldsymbol{\delta}\mathbf{r} = -\nabla\, p' + \rho_0 \mathbf{g}' + \rho' \mathbf{g}_0 \; . \tag{8.16}$$

Here the term in $\mathbf{v}_0$ is a small perturbation. Thus we can investigate its effect by means of perturbation analysis, as discussed in Section 5.5. Following equations (5.56) and (5.57) I write equation (8.16) as

$$\omega^2 \boldsymbol{\delta}\mathbf{r} = \mathcal{F}(\boldsymbol{\delta}\mathbf{r}) + \delta\mathcal{F}(\boldsymbol{\delta}\mathbf{r}) \; , \tag{8.17}$$

where

$$\delta\mathcal{F}(\boldsymbol{\delta}\mathbf{r}) = -2i\omega(\mathbf{v}_0 \cdot \nabla)\boldsymbol{\delta}\mathbf{r} \; . \tag{8.18}$$

It now follows from equation (5.73) and the definition of the inner product that the change in ω caused by the velocity field is, to first order,

$$\delta\omega = -i\frac{\int_V \rho_0 \boldsymbol{\delta}\mathbf{r}^* \cdot (\mathbf{v}_0 \cdot \nabla)\boldsymbol{\delta}\mathbf{r}\,\mathrm{d}V}{\int_V \rho_0 |\boldsymbol{\delta}\mathbf{r}|^2\,\mathrm{d}V} \; . \tag{8.19}$$

8.2 The effect of pure rotation

Now $\mathbf{v}_0$ is taken to correspond to a pure rotation, with angular velocity $\Omega = \Omega(r, \theta)$ that may depend on r and θ. I assume that the entire star is rotating around a common axis and choose this as the axis of the spherical polar coordinate system. Then

$$\mathbf{v}_0 = \Omega r \sin\theta \, \mathbf{a}_\phi = \boldsymbol{\Omega} \times \mathbf{r} \,, \tag{8.20}$$

where I have introduced the rotation vector

$$\boldsymbol{\Omega} = \Omega(\cos\theta \, \mathbf{a}_r - \sin\theta \, \mathbf{a}_\theta) \,. \tag{8.21}$$

We must now evaluate equation (8.19) for a normal mode of oscillation, and hence we have to consider the derivative in the direction of $\mathbf{v}_0$. From equation (4.30) the perturbations depend on ϕ as $\exp(im\phi)$; thus for a scalar quantity a

$$(\mathbf{v}_0 \cdot \nabla) a = \Omega r \sin\theta \frac{1}{r \sin\theta} \frac{\partial a}{\partial \phi} = im\Omega \, a \,. \tag{8.22}$$

For a vector $\mathbf{F}$ I use equation (4.10), and note that the directional derivatives of the coordinates of $\mathbf{F}$ can be found by using equation (8.22). The result is

$$(\mathbf{v}_0 \cdot \nabla)\mathbf{F} = im\Omega \mathbf{F} + \Omega[-F_\phi \sin\theta \, \mathbf{a}_r - F_\phi \cos\theta \, \mathbf{a}_\theta + (F_r \sin\theta + F_\theta \cos\theta) \, \mathbf{a}_\phi] \,. \tag{8.23}$$

This can also be written as

$$(\mathbf{v}_0 \cdot \nabla)\mathbf{F} = im\Omega \mathbf{F} + \boldsymbol{\Omega} \times \mathbf{F} \,. \tag{8.24}$$

Thus equation (8.16) becomes

$$-\omega^2 \rho_0 \boldsymbol{\delta}\mathbf{r} + 2m\omega\Omega\rho_0 \boldsymbol{\delta}\mathbf{r} - 2i\omega\rho_0 \boldsymbol{\Omega} \times \boldsymbol{\delta}\mathbf{r} = -\nabla p' + \rho_0 \mathbf{g}' + \rho' \mathbf{g}_0 \,. \tag{8.25}$$

In the case of a uniform rotation rate Ω this equation may be obtained much more simply. Here we may transform to a coordinate system rotating with the star, with coordinates $(r', \theta', \phi') = (r, \theta, \phi - \Omega t)$. In this system the dependence of the perturbations on ϕ' and t is as

$$\cos(m\phi' + m\Omega t - \omega t) = \cos(m\phi' - \omega' t) \,, \tag{8.26}$$

where $\omega' \equiv \omega - m\Omega$ (see also the simple analysis in the introduction to this chapter). To write down the equations of motion in the rotating system I note that here there is no term in the equilibrium velocity; however, we must add the term $-2\rho_0 \boldsymbol{\Omega} \times \boldsymbol{\delta}\mathbf{v}$ from the Coriolis force on the right-hand side. Using that to the required order of precision the velocity perturbation is $\boldsymbol{\delta}\mathbf{v} = -i\omega \boldsymbol{\delta}\mathbf{r}$, the result is

$$-\omega'^2 \rho_0 \boldsymbol{\delta}\mathbf{r} = -\nabla p' + \rho_0 \mathbf{g}' + \rho' \mathbf{g}_0 + 2i\omega\rho_0 \boldsymbol{\Omega} \times \boldsymbol{\delta}\mathbf{r} \,. \tag{8.27}$$

But this agrees with equation (8.25), if a term in Ω^2 is neglected.

In the general case of non-uniform rotation it might be argued that this relation would hold locally at any given point in the fluid, thus resulting again in equation (8.25). However, it is not clear (to me, at least) whether this is a consistent derivation of that relation, or whether it results from fortuitous cancellation of terms coming from the variation of Ω. In any case it allows a simple interpretation of the two terms in Ω in equation (8.25): the first term comes from the coordinate rotation, or equivalently from the advection of the rotating star relative to an observer in an inertial frame, and the second term comes from the Coriolis force.

We must now calculate the integral. By substituting $\boldsymbol{\delta}\mathbf{r}$, given by the complex form of equation (4.40), for $\mathbf{F}$ in equation (8.23) we obtain

$$\begin{aligned}(\mathbf{v}_0 \cdot \nabla)\boldsymbol{\delta}\mathbf{r} &= im\Omega\,\boldsymbol{\delta}\mathbf{r} \\ &+\sqrt{4\pi}\Omega\left[-\xi_\mathrm{h}\frac{\partial Y_l^m}{\partial\phi}\mathbf{a}_r - \xi_\mathrm{h}\frac{\cos\theta}{\sin\theta}\frac{\partial Y_l^m}{\partial\phi}\mathbf{a}_\theta + \left(\xi_r\sin\theta\, Y_l^m + \xi_\mathrm{h}\cos\theta\frac{\partial Y_l^m}{\partial\theta}\right)\mathbf{a}_\phi\right] .\end{aligned} \tag{8.28}$$

Thus

$$\begin{aligned}\tilde{R} &\equiv \int_V \rho_0\boldsymbol{\delta}\mathbf{r}^* \cdot (\mathbf{v}_0\cdot\nabla)\boldsymbol{\delta}\mathbf{r}\,\mathrm{d}V \\ &= im\int_V \rho_0\Omega|\boldsymbol{\delta}\mathbf{r}|^2\,\mathrm{d}V + 4\pi\int_V\rho_0\Omega\left[-\xi_r{}^*(Y_l^m)^*\xi_\mathrm{h}\frac{\partial Y_l^m}{\partial\phi} - |\xi_\mathrm{h}|^2\left(\frac{\partial Y_l^m}{\partial\theta}\right)^*\frac{\partial Y_l^m}{\partial\phi}\frac{\cos\theta}{\sin\theta}\right. \\ &\left.+\xi_\mathrm{h}^*\xi_r\left(\frac{\partial Y_l^m}{\partial\phi}\right)^* Y_l^m + |\xi_\mathrm{h}|^2\left(\frac{\partial Y_l^m}{\partial\phi}\right)^*\frac{\partial Y_l^m}{\partial\theta}\frac{\cos\theta}{\sin\theta}\right]\mathrm{d}V .\end{aligned} \tag{8.29}$$

Here Y_l^m is always multiplied by its complex conjugate, so that the ϕ-dependence cancels. Hence the integration over ϕ is trivial. It should be noticed also that all terms in the second integral in equation (8.29) contain the ϕ-derivative of Y_l^m or its complex conjugate, which is proportional to im. Thus $\tilde{R}$ contains im as a factor, and can be written, using equation (4.30), as

$$\tilde{R} = im\,8\pi^2 c_{lm}^2 R_{nlm} , \tag{8.30}$$

where

$$\begin{aligned}R_{nlm} &= \int_0^\pi \sin\theta\mathrm{d}\theta\int_0^R\left\{|\xi_r(r)|^2 P_l^m(\cos\theta)^2\right. \\ &+|\xi_\mathrm{h}(r)|^2\left[\left(\frac{\mathrm{d}P_l^m}{\mathrm{d}\theta}\right)^2 + \frac{m^2}{\sin^2\theta}P_l^m(\cos\theta)^2\right] \\ &-P_l^m(\cos\theta)^2\left[\xi_r{}^*(r)\xi_\mathrm{h}(r) + \xi_r(r)\xi_\mathrm{h}^*(r)\right] \\ &\left.-2P_l^m(\cos\theta)\frac{\mathrm{d}P_l^m}{\mathrm{d}\theta}\frac{\cos\theta}{\sin\theta}|\xi_\mathrm{h}(r)|^2\right\}\Omega(r,\theta)\rho_0(r)r^2\mathrm{d}r .\end{aligned} \tag{8.31}$$

Similarly, the denominator in equation (8.19) can be written as

$$\tilde{I} \equiv \int_V \rho_0|\boldsymbol{\delta}\mathbf{r}|^2\mathrm{d}V = 8\pi^2 c_{lm}^2 I_{nlm} , \tag{8.32}$$

where

$$\begin{aligned}I_{nlm} &= \int_0^\pi\sin\theta\mathrm{d}\theta\int_0^R\left\{|\xi_r(r)|^2P_l^m(\cos\theta)^2\right. \\ &\left.+|\xi_\mathrm{h}(r)|^2\left[\left(\frac{\mathrm{d}P_l^m}{\mathrm{d}\theta}\right)^2 + \frac{m^2}{\sin^2\theta}P_l^m(\cos\theta)^2\right]\right\}\rho_0(r)r^2\mathrm{d}r \\ &= \frac{2}{2l+1}\frac{(l+|m|)!}{(l-|m|)!}\int_0^R\left[|\xi_r|^2 + l(l+1)|\xi_\mathrm{h}|^2\right]\rho_0(r)r^2\mathrm{d}r\end{aligned} \tag{8.33}$$

[compare with equation (4.47)]. From equations (8.19), (8.30) and (8.32) we finally obtain the *rotational splitting*, *i.e.*, the perturbation in the frequencies caused by rotation, as

$$\delta\omega_{nlm} = m\frac{R_{nlm}}{I_{nlm}} \; . \tag{8.34}$$

This may obviously be written on the form

$$\delta\omega_{nlm} = m\int_0^R\int_0^\pi K_{nlm}(r,\theta)\Omega(r,\theta)r\mathrm{d}r\mathrm{d}\theta \; , \tag{8.35}$$

where the *kernel* K_{nlm} is defined by equations (8.31) and (8.33).

From equations (8.31) and (8.33), as well as the symmetry property of the Legendre function with respect to m (eq. A.8), it follows that R_{nlm}/I_{nlm} is an even function of m and hence that $\delta\omega_{nlm}$ is an odd function of m,

$$\delta\omega_{nl\,-m} = -\delta\omega_{nlm} \; . \tag{8.36}$$

Also, since $P_l^m(x)$ is either symmetrical or antisymmetrical around $x = 0$, the factor multiplying $\Omega(r,\theta)$ in equation (8.31) is symmetrical around the equator, $\theta = \pi/2$; thus

$$K_{nlm}(r, \pi-\theta) = K_{nlm}(r,\theta) \; . \tag{8.37}$$

This has the important consequence that the rotational frequency splitting is sensitive only to the part of Ω that is symmetrical around the equator.

Exercise 8.1:

Confirm the symmetry properties in equations (8.36) and (8.37).

The rotational splitting for a uniformly rotating star was first obtained by Cowling & Newing (1949) and Ledoux (1949). The general case, as presented here, was considered by Hansen, Cox & van Horn (1977) and Gough (1981).

8.3 Splitting for spherically symmetric rotation

To proceed we must make an explicit assumption about the variation of Ω with θ. For simplicity I shall assume first that Ω is independent of θ. In fact, as mentioned earlier, the solar surface rotation depends on θ; however, the assumption of θ-independent rotation can be regarded as the first term in an expansion of Ω, say, in terms of powers of $\cos\theta$. In this case the integrals over θ in equation (8.31) only involve Legendre functions and may be evaluated analytically. Two of the terms require a little care. One contains

$$\begin{aligned}\int_0^\pi P_l^m(\cos\theta)\frac{\mathrm{d}P_l^m}{\mathrm{d}\theta}\frac{\cos\theta}{\sin\theta}\sin\theta\mathrm{d}\theta &= -\int_{-1}^1 P_l^m(x)\frac{\mathrm{d}P_l^m}{\mathrm{d}x}x\mathrm{d}x \\ &= -\left.\frac{1}{2}xP_l^m(x)^2\right|_{-1}^1 + \frac{1}{2}\int_{-1}^1 P_l^m(x)^2\mathrm{d}x \; ,\end{aligned} \tag{8.38}$$

and here the integrated term vanishes, as $P_l^m(x)$ is either symmetrical or anti-symmetrical in $x = \cos\theta$. The other non-trivial integral, which was already encountered in the evaluation of I_{nlm}, is

$$\int_0^\pi \left[\left(\frac{\mathrm{d}P_l^m}{\mathrm{d}\theta} \right)^2 + \frac{m^2}{\sin^2\theta} P_l^m(\cos\theta)^2 \right] \sin\theta \mathrm{d}\theta \tag{8.39}$$
$$= -\int_0^\pi P_l^m(\cos\theta) \left[\frac{\mathrm{d}}{\mathrm{d}\theta} \left(\sin\theta \frac{\mathrm{d}P_l^m}{\mathrm{d}\theta} \right) - \frac{m^2}{\sin\theta} P_l^m(\cos\theta) \right] \mathrm{d}\theta$$
$$= L^2 \int_{-1}^1 P_l^m(x)^2 \mathrm{d}x \ ,$$

by using that P_l^m satisfies equation (4.26). As usual, I have introduced $L^2 \equiv l(l+1)$. For adiabatic oscillations we can take ξ_r and ξ_h to be real. Thus, from equation (8.31), (8.33) and (8.34), we finally obtain for the rotational splitting

$$\delta\omega_{nlm} = m \frac{\int_0^R \Omega(r) \left({\xi_r}^2 + L^2\xi_\mathrm{h}^2 - 2\xi_r\xi_\mathrm{h} - \xi_\mathrm{h}^2 \right) r^2 \rho \mathrm{d}r}{\int_0^R \left({\xi_r}^2 + L^2\xi_\mathrm{h}^2 \right) r^2 \rho \mathrm{d}r} \ , \tag{8.40}$$

where I have dropped the subscript "0" on ρ. It should be noticed that the integrands in equation (8.40) are given solely in terms of ξ_r, ξ_h and l, and therefore are independent of m. Hence in the case of spherically symmetric rotation the rotational splitting is proportional to m.

It is convenient to write equation (8.40) as

$$\delta\omega_{nlm} = m\beta_{nl} \int_0^R K_{nl}(r)\Omega(r) \mathrm{d}r \ , \tag{8.41}$$

where

$$K_{nl} = \frac{\left({\xi_r}^2 + L^2\xi_\mathrm{h}^2 - 2\xi_r\xi_\mathrm{h} - \xi_\mathrm{h}^2 \right) r^2 \rho}{\int_0^R \left({\xi_r}^2 + L^2\xi_\mathrm{h}^2 - 2\xi_r\xi_\mathrm{h} - \xi_\mathrm{h}^2 \right) r^2 \rho \mathrm{d}r} \ , \tag{8.42}$$

and

$$\beta_{nl} = \frac{\int_0^R \left({\xi_r}^2 + L^2\xi_\mathrm{h}^2 - 2\xi_r\xi_\mathrm{h} - \xi_\mathrm{h}^2 \right) r^2 \rho \mathrm{d}r}{\int_0^R \left({\xi_r}^2 + L^2\xi_\mathrm{h}^2 \right) r^2 \rho \mathrm{d}r} \ . \tag{8.43}$$

By using this definition we ensure that the *rotational kernel* K_{nl} is unimodular, *i.e.*,

$$\int_0^R K_{nl}(r) \mathrm{d}r = 1 \ . \tag{8.44}$$

Hence for uniform rotation, where $\Omega = \Omega_\mathrm{s}$ is constant,

$$\delta\omega_{nlm} = m\beta_{nl}\Omega_\mathrm{s} \ . \tag{8.45}$$

In this case the effect of rotation is completely given by the constant β_{nl}. For high-order or high-degree p modes the terms in ${\xi_r}^2$ and $L^2\xi_\mathrm{h}^2$ dominate; as shown in Figure 8.1, β_{nl} is then close to one. Thus the rotational splitting between adjacent m-values is given approximately by the rotation rate. Physically, the neglected terms in equation (8.43) arise

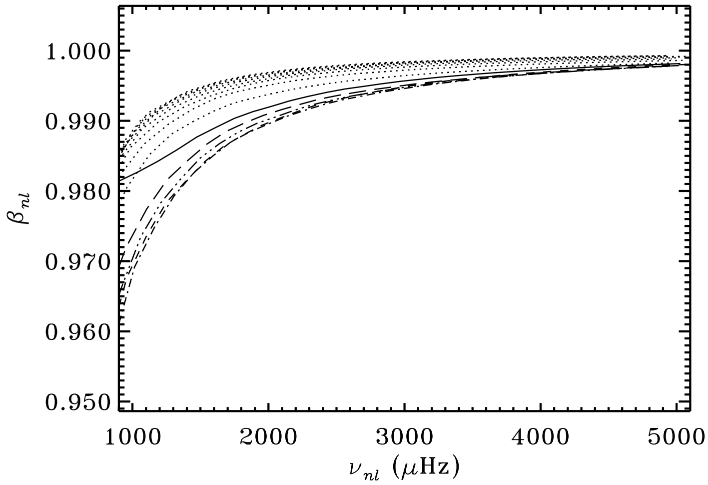

Figure 8.1: Coefficients β_{nl} for acoustic modes in a normal solar model. Points corresponding to fixed l have been connected, according to the following line styles: $l = 1$: ————; $l = 2$: ------------; $l = 3$: —·—·—·—; $l = 4$: —···—···—···; $l = 5$: – – – – – – –; $l = 10, 15, \ldots, 50$: ················ (with β_{nl} increasing with l).

from the Coriolis force; thus rotational splitting for p modes is dominated by advection. For high-order g modes, on the other hand, we can neglect the terms containing ξ_r, so that

$$\beta_{nl} \simeq 1 - \frac{1}{L^2} \; . \tag{8.46}$$

In particular, the splitting of high-order g modes of degree 1 is only *half* the rotation rate.

Returning to the case where Ω depends on r, it should be noted that the integral in equation (8.41) provides a weighted average of $\Omega(r)$. For high-order p modes we can use the asymptotic behaviour of the eigenfunctions to obtain

$$\delta\omega_{nlm} \simeq m \frac{\displaystyle\int_{r_\mathrm{t}}^R \left(1 - \frac{L^2 c^2}{r^2 \omega_{nl}^2}\right)^{-1/2} \Omega(r) \, \frac{\mathrm{d}r}{c}}{\displaystyle\int_{r_\mathrm{t}}^R \left(1 - \frac{L^2 c^2}{r^2 \omega_{nl}^2}\right)^{-1/2} \frac{\mathrm{d}r}{c}} \simeq m \, \frac{\displaystyle\int_{r_\mathrm{t}}^R \Omega(r) \, \frac{\mathrm{d}r}{c}}{\displaystyle\int_{r_\mathrm{t}}^R \frac{\mathrm{d}r}{c}} \; , \tag{8.47}$$

where in the last equality I crudely approximated $(1 - L^2 c^2 / r^2 \omega^2)$ by 1. Note that the last equality corresponds to neglecting the terms in ξ_h in equation (8.42) and using that, according to equation (7.34), $\xi_r \sim (\rho c)^{-1/2} r^{-1}$ outside the turning point; in this approximation we obtain the intuitively appealing result that the rotational splitting is an average of the rotation rate, weighted by the sound travel time in the radial direction. The first, more accurate expression can also be obtained from ray theory (Gough 1984). In fact, it is straightforward to show that the weight given to $\Omega(r)$ is simply the sound-travel time, corresponding to the radial distance dr, along a ray; this evidently becomes infinite at the lower turning point. It should be noted that the first part of equation (8.47) may also be written as

$$S_{nl}\delta\omega_{nlm} \simeq m \int_{r_\mathrm{t}}^{R} \left(1 - \frac{c^2 L^2}{\omega_{nl}^2 r^2}\right)^{-1/2} \Omega(r) \frac{\mathrm{d}r}{c} \; , \tag{8.48}$$

in complete analogy with equation (7.145) for the frequency change resulting from a change in the sound speed; here I neglected the term $\pi \mathrm{d}\alpha/\mathrm{d}\omega$ in equation (7.146) which in general is small.

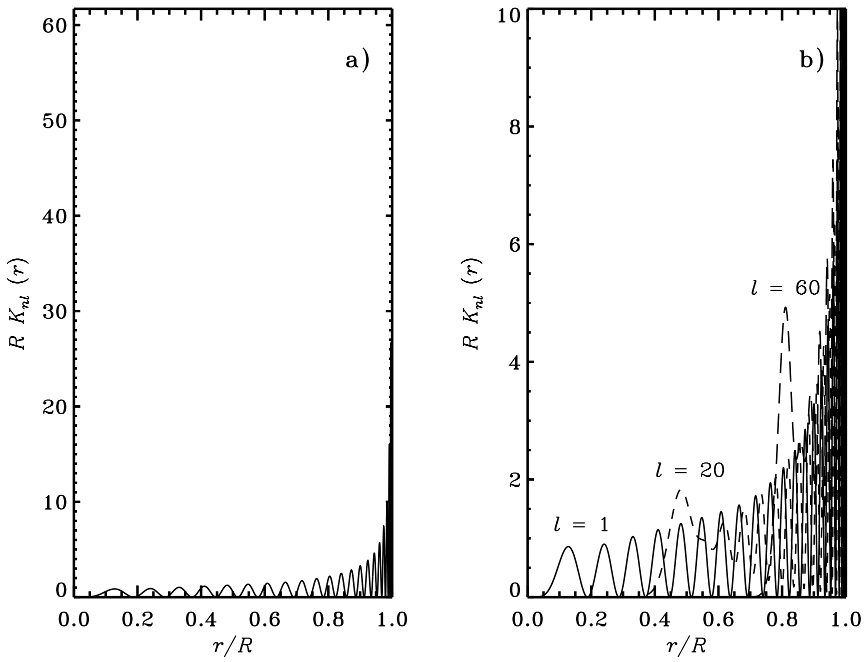

Figure 8.2: Kernels K_{nl} for the frequency splitting caused by spherically symmetric rotation (*cf.* eq. 8.42). In a) is plotted $RK_{nl}(r)$ for a mode with $l = 1$, $n = 22$ and $\nu = 3239\,\mu$Hz. The maximum value of $RK_{nl}(r)$ is 62. In b) is shown the same mode, on an expanded vertical scale, (————) together with the modes $l = 20$, $n = 17$, $\nu = 3375\,\mu$Hz (------------), and $l = 60$, $n = 10$, $\nu = 3234\,\mu$Hz (– – – – – – –). Notice that the kernels almost vanish inside the turning-point radius r_t, and that there is an accumulation just outside the turning point.

Figure 8.2 shows a few kernels for the case of spherically symmetric rotation, for high-order p modes. The strong increase towards the solar surface, which is also implicit in equation (8.47), is evident. Also, the kernels clearly get very small beneath the turning point, but are locally enhanced just above it. This effect arises from the term in ξ_h in equation (8.42); physically it corresponds to the fact that the waves travel approximately horizontally in this region, and hence spend a relatively long time there, as also indicated by the integrable singularity at $r = r_{\rm t}$ in equation (8.47).

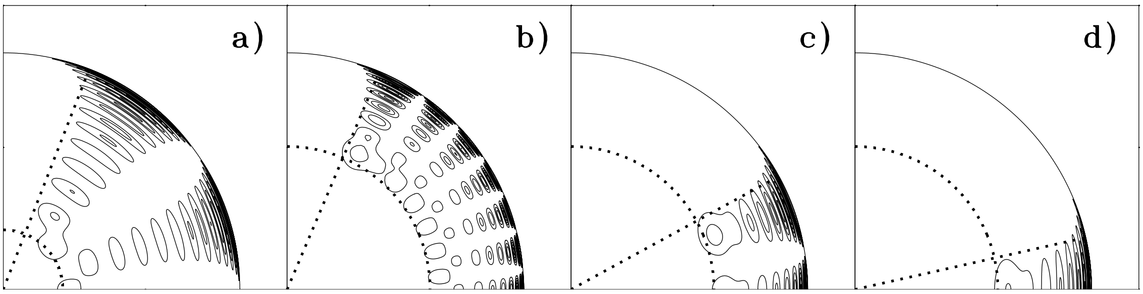

Figure 8.3: Contour plots of rotational kernels K_{nlm} in a solar quadrant. The modes all have frequencies near 2 mHz; the following pairs of (l, m) are included: a) $(5, 2)$; b) $(20, 8)$; c) $(20, 17)$; and d) $(20, 20)$. The dotted circles mark the locations of the lower radial turning point $r_{\rm t}$ and the dotted lines show the latitudinal turning points, at co-latitude Θ, defined by $\sin\Theta = m/L$.

8.4 General rotation laws

In the general case, where Ω depends on both r and θ, the rotational splitting may be computed from equations (8.31), (8.33) and (8.34), by evaluating the two-dimensional integral in equation (8.31). This integral is in general m-dependent, and so the splitting is no longer a linear function of m. Selected examples of the resulting kernels $K_{nlm}(r, \theta)$ (*cf.* eq. 8.35) are illustrated in Figure 8.3.

To illustrate the properties of the splitting, it is instructive to rewrite equation (8.31) for R_{nlm}, using integration by parts:

$$\begin{aligned} R_{nlm} &= \int_0^\pi {\rm d}\theta \int_0^R P_l^m(\cos\theta)^2 \left\{ \left[{\xi_r}^2 + (L^2 - 1)\xi_{\rm h}^2 - 2\xi_r\xi_{\rm h} \right] \sin\theta\, \Omega(r, \theta) \right. \\ & \qquad \left. + \xi_{\rm h}^2 \left(\frac{3}{2} \cos\theta \frac{\partial\Omega}{\partial\theta} + \frac{1}{2} \sin\theta \frac{\partial^2\Omega}{\partial\theta^2} \right) \right\} r^2 \rho \, {\rm d}r \end{aligned} \qquad (8.49)$$

(Cuypers 1980). I consider again the case of high-order p modes; here the terms in ${\xi_r}^2$ and $L^2\xi_{\rm h}^2$ dominate, and consequently

$$\delta\omega_{nlm} \simeq m \frac{\int_0^\pi \sin\theta\, [P_l^m(\cos\theta)]^2 \int_0^R \Omega(r, \theta)[\xi_r(r)^2 + L^2\xi_{\rm h}(r)^2] r^2 \rho \, {\rm d}r{\rm d}\theta}{\int_0^\pi \sin\theta\, [P_l^m(\cos\theta)]^2 \, {\rm d}\theta \int_0^R [\xi_r(r)^2 + L^2\xi_{\rm h}(r)^2] r^2 \rho \, {\rm d}r} \; . \qquad (8.50)$$

Hence, the splitting is simply an average of the angular velocity $\Omega(r, \theta)$, weighted by $r^2\rho[\xi_r(r)^2 + L^2\xi_{\rm h}(r)^2] P_l^m(\cos\theta)^2$. Approximating the eigenfunction as in the derivation of equation (8.47) and using, furthermore, an asymptotic approximation to P_l^m, this may be written as

$$\delta\omega_{nlm} \simeq \tag{8.51}$$
$$m \frac{\int_{-\cos\Theta}^{\cos\Theta} \left(\cos^2\Theta - \cos^2\theta\right)^{-1/2} \int_{r_{\rm t}}^{R} \left(1 - \frac{L^2 c^2}{r^2\omega^2}\right)^{-1/2} \Omega(r,\theta)\, \frac{{\rm d}r}{c}\, {\rm d}(\cos\theta)}{\pi \int_{r_{\rm t}}^{R} \left(1 - \frac{L^2 c^2}{r^2\omega^2}\right)^{-1/2} \frac{{\rm d}r}{c}} ,$$

where $\Theta = \sin^{-1}(m/L)$ (Gough and Thompson 1990, 1991; Gough 1991). The asymptotic approximation to P_l^m shows that a given spherical harmonic is confined essentially to the latitude band between $\pm\Theta$; within this region P_l^m oscillates as a function of θ, whereas at higher latitudes it decreases exponentially. The variation of the extent of the P_l^m with m/L allows resolution of the latitudinal variation of the angular velocity, in much the same way as the variation of the depth of penetration with ω/L allows resolution of the variation with radius. In particular, with increasing l the sectoral modes (with $l = |m|$) get increasingly confined towards the equator (see also Figure 2.1). Thus, the rotational splitting of sectoral modes provides a measure of the solar equatorial angular velocity.

Exercise 8.2:

Confirm the statements about the oscillatory properties of the spherical harmonics, by analyzing equation (4.26) satisfied by the Legendre function.

To study the splitting without making the asymptotic approximation, it is convenient to consider a parameterized representation of $\Omega(r,\theta)$. To illustrate the principle, we may consider an expansion on the form

$$\Omega(r,\theta) = \sum_{s=0}^{s_{\rm max}} \Omega_s(r) \cos^{2s}\theta \; . \tag{8.52}$$

Then the integrals over θ can be evaluated analytically, in much the same way as the derivation of equation (8.40), and the rotational splitting becomes (see also Cuypers 1980)

$$\delta\omega_{nlm} = m \sum_{s=0}^{s_{\rm max}} \int_0^R K_{nlms}(r)\Omega_s(r){\rm d}r \; . \tag{8.53}$$

Here

$$\begin{aligned} K_{nlms}(r) \;=\; & \rho r^2 I_{nl}^{-1} \left\{ \left[(\xi_r - \xi_{\rm h})^2 + (L^2 - 2s^2 - 3s - 2)\xi_{\rm h}^2 \right] Q_{lms} \right. \\ & \left. + s(2s-1)\xi_{\rm h}^2 Q_{lms-1} \right\} \; , \end{aligned} \tag{8.54}$$

where

$$I_{nl} = \int_0^R \rho r^2 \left({\xi_r}^2 + L^2 \xi_{\rm h}^2 \right) {\rm d}r \; , \tag{8.55}$$

and

$$Q_{lms} = \frac{2l+1}{2} \frac{(l-|m|)!}{(l+|m|)!} \int_{-1}^{1} x^{2s} [P_l^m(x)]^2 {\rm d}x \; . \tag{8.56}$$

For spherically symmetric rotation, $\Omega_s = 0$ for $s > 0$. Since $Q_{lm0} = 1$, the kernel $K_{nlm0}(r) = \beta_{nl} K_{nl}(r)$ is independent of m, and the splitting is uniform in m. Thus we recover the results of Section 8.3.

The factor Q_{lms} is a polynomial in m^2 of degree s; thus, in accordance with equation (8.36), $\delta\omega_{nlm}$ is a polynomial of odd powers of m. Up to $s = 2$ explicit expressions for the Q_{lms} are

$$Q_{lm1} = \frac{2L^2 - 2m^2 - 1}{4L^2 - 3} , \tag{8.57}$$

and

$$Q_{lm2} = R_{l+1}^m \left(R_{l+2}^m + R_{l+1}^m + R_l^m \right) + R_l^m \left(R_{l+1}^m + R_l^m + R_{l-1}^m \right) , \tag{8.58}$$

where

$$R_l^m = \frac{l^2 - m^2}{4l^2 - 1} . \tag{8.59}$$

Hence equation (8.53) leads to an expansion of the rotational splitting in odd powers of m, with expansion coefficients that are related to the expansion functions $\Omega_s(r)$. As discussed by Brown *et al.* (1989), this forms the basis for an inversion to determine the Ω_s, and hence to estimate the rotation rate as a function of r and θ.

The choice of expansion

$$\Omega(r, \theta) = \sum_{s=0}^{s_{\max}} \Omega_s(r) \psi_{2s}(\cos\theta) , \tag{8.60}$$

for $\Omega(r, \theta)$, and of the expansion for $\delta\omega_{nlm}$, are clearly not unique. In particular, it was pointed out by Ritzwoller & Lavely (1991) that a more suitable expansion of the rotational splitting could be obtained in terms of Clebsch-Gordon coefficients. This is equivalent to the odd terms in the expansion (2.43) of the m-dependence of the frequencies in terms of the polynomials $\mathcal{P}_j^{(l)}(m)$. Choosing also expansion functions for the rotation rate $\Omega(r, \theta)$ as

$$\psi_{2s}(\cos\theta) = -(\sin\theta)^{-1} P_{2s+1}^1(\cos\theta) \tag{8.61}$$

(*e.g.* Ritzwoller & Lavely 1991; Pijpers 1997), the relations decouple such that each expansion coefficient for the splitting is related to a single expansion function for the angular velocity:

$$2\pi a_{2s+1}(n, l) = \int_0^R K_{nls}^s(r) \Omega_s(r) \mathrm{d}r \tag{8.62}$$

(see also Schou *et al.* 1998). This forms a convenient basis for the so-called 1.5-dimensional inversions for $\Omega(r, \theta)$ (*cf.* Section 9.1.2).

It should be noted that, in general, averaging or expansion of the observed frequencies may involve loss of information; for the purpose of inversion it is, in principle, preferable to work directly in terms of the observed frequencies. In fact, as mentioned in Chapter 9 a direct inversion of the frequency splittings $\delta\omega_{nlm}$ for individual m, in terms of a discretized representation of $\Omega(r, \theta)$ on a grid in r and θ, is now computationally feasible (see also Schou, Christensen-Dalsgaard & Thompson 1994). On the other hand, by suitably combining the frequencies before inversion, the computational effort required may be greatly reduced. Furthermore, it is often the case that the quality of the observations does not allow a complete determination of the individual frequencies; in that case inversion has to be based on expansions such as the one given in equation (2.43).